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Sliding mode control for a generalization of the Caginalp phase-field system

Controllo di tipo sliding-mode per un sistema di campo di fase che estende il modello di Caginalp

Relatore:
Ch.mo Prof.
Pierluigi Colli

Candidato:
Davide Manini
Matr: 461345

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Abstract

Questa tesi è dedicata allo studio dell'esistenza di *sliding modes* in un modello per le transizioni di fase di tipo *phase field*. Le sue incognite sono lo spostamento termico w (di fatto una primitiva rispetto al tempo della temperatura) e il parametro di fase φ. Le equazioni del modello (la prima delle quali estende il sistema di Caginalp),

$$(w_t + l\varphi)_t - \kappa \triangle w_t - \tau \triangle w + \rho \operatorname{Sign}(w_t + \alpha\varphi - \eta^*) \ni f,$$
$$\varphi_t - \triangle\varphi + F'(\varphi) = \gamma w_t,$$

sono abbinate alle condizioni al bordo di Neumann omogenee e alle condizioni iniziali per w, $\partial_t w$ e φ. Il sistema evolve all'interno di un dominio regolare e limitato $\Omega \subset \mathbb{R}^N$. Le soluzioni del sistema sono condizionate, oltre che dai dati iniziali, dai parametri positivi l, κ, τ, ρ, α e γ, dal termine di sorgente f, dalla funzione bersaglio η^* indipendente dal tempo e dal potenziale $F(\varphi)$: quest'ultimo è caratterizzato dalla presenza di due minimi, uno per $\varphi > 0$ e uno per $\varphi < 0$, che rappresentano le due fasi termodinamiche.

In questa tesi dimostriamo che per opportuni valori di ρ esiste un tempo T^* dopo il quale le soluzioni soddisfano

$$w_t + \alpha\varphi = \eta^*,$$

cioè si annulla l'argomento dell'operatore Sign. Proprio questo termine Sign era stato introdotto allo scopo di realizzare gli sliding modes [2] nel sistema originale di Caginalp [6]. La novità del presente lavoro risiede nell'estendere l'analisi delle sliding modes a un modello più complesso, che tiene anche conto della memoria evolutiva del sistema. L'esistenza delle soluzioni e degli sliding modes è dimostrata in generale, mentre la dipendenza continua dai dati iniziali e l'unicità della soluzione sono dedotte nel caso speciale $l = \alpha$.

Le tecniche principali usate in questo lavoro sono la teoria degli operatori massimali monotoni in spazi di Hilbert, in particolare utilizzando le regolarizzazioni di Yosida e di Moreau–Yosida, e l'approssimazione di Faedo–Galerkin per lo studio di equazioni differenziali alle derivate parziali.

Contents

Chapter 1

Introduction

For many years, sliding mode control (SMC) has been recognized as one of the best approaches for the design of robust controllers for nonlinear dynamical systems. Nowadays, SMC is considered a standard tool for the regulation of time-evolving systems in finite dimension [3, 12, 15, 28, 30].

The design of feedback control systems with sliding modes involves the construction of suitable control functions enforcing motion along a given manifold of lower dimension, called *sliding manifold*. The main idea is (i) to identify this manifold where the control target is fulfilled and such that the original system restricted to this sliding manifold has a desired behavior; (ii) to act on the system through a suitable control term in order to constrain the evolution on it. This new term forces the trajectories of the system to reach the sliding manifold and maintains them along it.

Sliding mode controls feature robustness with respect to unmodelled dynamics and insensitivity to external disturbances. At the same time they are relatively easy to design. For these reasons, in the last years there has been a growing interest in bringing these methods for finite-dimensional systems described by ODEs [19, 22, 23] to the realm of PDEs. While certain early works going in this direction [21, 23, 24] deal with particular classes of PDEs, the theoretical development in a general Hilbert space setting has gained attention only in the last years [10, 20, 29].

In this thesis, the considered system describes the spatial and time fluctuations close to a phase transition. In order to take into account the effects of phase dissipation, Caginalp introduced [6] a phase-field system consisting of the following equations

$$(\vartheta + l\varphi)_t - \kappa\,\triangle\vartheta = f, \qquad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$\varphi_t - \triangle\varphi + F'(\varphi) = \gamma\vartheta, \qquad \text{in } \Omega \times (0, T). \tag{1.2}$$

$\Omega \subset \mathbb{R}^N$ represents the spatial domain where the evolution takes place and

$T > 0$ is the final time of the evolution. While the case $N = 3$ is the one of physical interest, we will carry out our analysis for all N. A usual choice, at least for the phase variable φ, is to complement the equations with standard homogeneous Neumann boundary conditions $\partial_n \varphi = \partial_n \vartheta = 0$ on $\partial\Omega$, plus the initial conditions $\vartheta(\cdot, 0) = \vartheta_0$ and $\varphi(\cdot, 0) = \varphi_0$.

The variable ϑ represents the relative temperature, i.e. the difference between the actual temperature and the fixed critical temperature for the phase transition. The variable φ has the meaning of a phase parameter: $\varphi < 0$ indicates one of the two phases; $\varphi > 0$ indicates the other phase. $\varphi(x, t) = 0$ usually indicates that position x is at the interface between the two phases at time t. F' is the derivative of a double-well potential F. A few examples for the double-well potential F are

$$F(r) = \frac{1}{4}(r^2 - 1)^2, \tag{1.3}$$

$$F(r) = \begin{cases} -c_0 r^2, & \text{if } |r| \leq 1, \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.4}$$

$$F(r) = \begin{cases} (1+r)\log(1+r) + (1-r)\log(1-r) - (c_0+1)r^2, & \text{if } |r| < 1, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.5}$$

where $c_0 \in \mathbb{R}$, $c_0 > 0$.

The physical equations originating the system (1.1)–(1.2) are[1]

$$\partial_t e + \operatorname{div} \mathbf{q} = \tilde{f}, \tag{1.6}$$

$$\partial_t \varphi + \frac{\delta \mathcal{F}}{\delta \varphi} = 0, \tag{1.7}$$

where e denotes the internal energy, $\mathbf{q}$ the thermal flux, and $\tilde{f}$ the heat source. The term $\frac{\delta \mathcal{F}}{\delta \varphi}$ represents the variational derivative with respect to φ of the following functional

$$\mathcal{F}(\vartheta, \varphi) = \int_\Omega \left(-\frac{c_0}{2}\vartheta^2 - \gamma\vartheta\varphi + F(\varphi) + \frac{1}{2}|\nabla\varphi|^2 \right), \tag{1.8}$$

where the constants c_0 and γ represent the specific heat and the latent heat coefficient, respectively. Note that the term $-\gamma\vartheta\varphi$ favors states having concordant relative temperature and phase variable. The internal energy e can

[1] Please note that in this thesis we will use both the notations p_t and $\partial_t p$ to denote the derivative of a function p.

be derived from the functional $\mathcal{F}$, taking minus its variational derivative with respect to ϑ, i.e.,

$$e = -\frac{\delta \mathcal{F}}{\delta \vartheta} = c_0 \vartheta + \gamma \varphi. \tag{1.9}$$

Equation (1.7) yields equation (1.2) by standard variational derivative taking into account the homogeneous Neumann condition for φ. We set $l := \gamma / c_0$ and $f := \tilde{f} / c_0$. If we assume the classic Fourier law

$$\mathbf{q} = -c_0 \kappa \nabla \vartheta, \tag{1.10}$$

equation (1.6) yields (1.1). The homogeneous Neumann condition for ϑ follows from the no-flux condition $\mathbf{q} \cdot \mathbf{n} = 0$ on the boundary of Ω.

A sliding-mode analysis has been carried out recently for the system described by the equations (1.1)–(1.2) [2]. In the quoted paper three cases are taken into consideration (labeled as Problem A–C). In Problem A, the sliding manifold is given by a linear constraint between ϑ and φ; in Problems B and C the phase φ is forced to reach a prescribed phase distribution φ^*. While in Problems A and B the control law is non-local in the spatial variable, in Problem C the control term is fully local.

In the present thesis we carry out a similar sliding-mode analysis, for modified equations where the Fourier law (1.10) is generalized in the light of the works by Green and Naghdi [16–18] and (more recently) by Podio-Guidugli [25] on thermodynamics. These papers introduced the notion of *thermal displacement*, which is a primitive of the temperature, i.e.

$$w(x,t) = w_0(x) + \int_0^t \vartheta(x,s) \mathrm{d}s, \tag{1.11}$$

where w_0 represents a given datum accounting for a possible previous thermal history of the phenomenon. Making use of this new variable, these authors proposed three theories for heat transmission labeled as type I–III. Type I theory, after suitable linearization, yields the standard Fourier law

$$\mathbf{q} = -c_0 \kappa \nabla w_t \qquad \text{(type I)}, \tag{1.12}$$

which has been studied extensively. Linearized versions of type II and III give the following heat-conduction laws

$$\mathbf{q} = -c_0 \tau \nabla w \qquad \text{(type II)}, \tag{1.13}$$

$$\mathbf{q} = -c_0 \kappa \nabla w_t - c_0 \tau \nabla w \qquad \text{(type III)}. \tag{1.14}$$

It is important to note that the thermal displacement w becomes necessary to describe type II and III laws, whereas type I law can be described just in terms of the temperature $\vartheta = \partial_t w$. The role of the primitive w in type II and III theories is to account for the past thermal history of the heat-conducting body.

This thesis focuses on the most general type III theory. In type III theory, the special $\tau = 0$ case reduces to standard type I theory; $\kappa = 0$ yields type II theory. Equation (1.6), along with type III law (1.14), leads to this formulation

$$(w_t + l\varphi)_t - \kappa \triangle w_t - \tau \triangle w = f, \qquad \text{in } \Omega \times (0,T). \tag{1.15}$$

Equation (1.2) with the substitution $\vartheta = \partial_t w$ becomes

$$\varphi_t - \triangle\varphi + F'(\varphi) = \gamma w_t, \qquad \text{in } \Omega \times (0,T). \tag{1.16}$$

The no-flux condition $\mathbf{q}\cdot\mathbf{n} = 0$ generates the homogeneous Neumann boundary condition $\partial_n w = 0$. For the system (1.15)–(1.16), well-posedness, asymptotic analysis, and convergence of the solutions as $\tau \to 0$ to the solution of the original Caginalp system (1.1)–(1.2) has been carried out in [7, 8].

In order to enable the SMC in the system above, we add a feedback term in equation (1.15) which forces the solutions $(w(t), \varphi(t))$ to reach the sliding manifold. We adopt the following linear condition connecting w and φ

$$\partial_t w(t) + \alpha\varphi(t) = \eta^*, \tag{1.17}$$

to describe the sliding manifold. In (1.17), α is a real positive constant and η^* a prescribed function independent of time. The feedback term we add to the left-hand side of equation (1.15) is

$$\rho\,\mathrm{Sign}(w_t + \alpha\varphi - \eta^*), \tag{1.18}$$

where Sign is the sign operator acting in the Hilbert space $L^2(\Omega)$, namely $\mathrm{Sign}(v) = v/\|v\|_{L^2(\Omega)}$ if $v \neq 0$, while Sign(0) gives the closed unit ball of $L^2(\Omega)$. In view of the above specification, let us emphasize that the control law is highly non-local in spatial variable, i.e. the value of the feedback term at (x,t) depends on $(w(\cdot,t), \varphi(\cdot,t))$ and not only on $(w(x,t), \varphi(x,t))$. The sliding-mode parameter $\rho > 0$ represents the strength of the control law and it plays a central role in this kind of analysis. Accordingly, we will highlight the dependence on ρ in all our estimates. The linear condition (1.17) as well as the choice of the sign operator in $L^2(\Omega)$ corresponds to the Problem (A) in [2] for the special case studied there. However, with respect to the arguments used in [2], here we adopt a slightly different approach, based on the simplification of the auxiliary lemma and the treatment of the difficult part in the proof of the theorem.

This thesis is organized as follows. The next chapter details the common notation and the considered system of equations; it also reports the precise results, i.e., the theorems for existence, uniqueness, and regularity of the solutions, a theorem for their continuous dependence on the initial data and finally the theorem ensuring the existence of sliding modes. The following chapters are devoted to the proofs. Let us notice here that the continuous dependence and the uniqueness results hold under the special condition $l = \alpha$, whereas for all other results this assumption is not required.

Chapter 2

Common notation and results

In this chapter, we introduce common notation, we present the problem that we will solve, as well as the results concerning well-posedness of the problem and regarding SMC. Moreover, a few technical tools are recalled.

First of all, we require for $\Omega \subset \mathbb{R}^N$ to be an open, bounded, smooth set. Γ and ∂_n represent the boundary of Ω and the outward normal derivative on Γ, respectively. We set $Q_t = \Omega \times (0,t)$ for $t \in (0,T]$ and $Q = Q_T$.

In the sequel, we will make use of techniques of convex analysis, so we split $F = \widehat{\beta} + \widehat{\pi}$, requiring that

$$\widehat{\beta} : \mathbb{R} \to [0,+\infty] \text{ is convex, proper, l.s.c. with } \widehat{\beta}(0) = 0, \tag{2.1}$$
$$\widehat{\pi} : \mathbb{R} \to \mathbb{R} \text{ is } C^1 \text{ and } \widehat{\pi}' \text{ is Lipschitz-continuous.} \tag{2.2}$$

We define β and π as the subdifferential [26, § 23] of $\widehat{\beta}$ and the derivative of $\widehat{\pi}$, respectively. It turns out that β is a maximal monotone graph of $\mathbb{R}^2$, such that $0 \in \beta(0)$. We indicate with $\beta^\circ(r)$ the element of $\beta(r)$ having minimum modulus.

We make the following assumptions on the data of the problem

$$\kappa, \tau, \gamma, l, \alpha \in (0,+\infty), \tag{2.3}$$
$$f \in L^2(Q). \tag{2.4}$$

We introduce the following Hilbert spaces

$$H := L^2(\Omega), \qquad V := H^1(\Omega), \qquad W := \{v \in H^2(\Omega) : \partial_n v = 0\}.$$

On H and V we put the standard Hilbert norm, while we endow W with the norm $\|u\|_W^2 = \|u\|_H^2 + \|\triangle u\|_H^2$, which is equivalent to the norm $\|\cdot\|_{H^2(\Omega)}$, by the smooth boundary condition and elliptic regularity (see e.g. [5, § 9.6] or [14, § 6.3]). The norm $\|\cdot\|_H$ will also denote the norm of the space $H^N =$

$L^2(\Omega;\mathbb{R}^N)$. The scalar product of H and H^N will be denoted with $(\cdot,\cdot)$. Throughout this work, the canonically isomorphic spaces $L^2(0,T;H)$ and $L^2(Q)$ will be identified. We define the sign operator for the Hilbert space H as the subdifferential of the norm $\|\cdot\|_H$, namely

$$\mathrm{Sign}(v) = \begin{cases} \dfrac{v}{\|v\|_H} & \text{if } v \neq 0, \\ B_H & \text{if } v = 0, \end{cases}$$

where B_H is the closed unit ball of H.

Remark 2.1. The maximal monotone graph β induce a natural maximal monotone operator β_Ω on H

$$\beta_\Omega(v) = \{w \in H : \, w(x) \in \beta(v(x)) \text{ for a.a. } x \in \Omega\},$$

for $v \in H$. In the same way we can define a monotone operator β_Q on $L^2(Q)$. At this point, we can repeat this operation, defining the operator $\beta_{T,H}$ on $L^2(0,T;H)$

$$\beta_{T,H}(v) = \{w \in L^2(0,T;H) : \, w(t) \in \beta_\Omega(v(t)) \text{ for a.a. } t \in (0,T)\},$$

for $v \in L^2(0,T;H)$. One can prove that β_Q and $\beta_{T,H}$ are the same operator up to the canonical isomorphism between $L^2(Q)$ and $L^2(0,T;H)$. For the sake of clarity, we will use only the symbol β to indicate all these operators. A similar remark can be done for the operator Sign.

The regularity hypotheses for the target function η^* and the initial data are

$$\eta^* \in W, \tag{2.5}$$

$$\vartheta_0 \in V, \qquad w_0 \in W, \qquad \varphi_0 \in V, \qquad \widehat{\beta}(\varphi_0) \in L^1(\Omega). \tag{2.6}$$

A solution is a quadruplet (w,φ,ξ,σ), for which we require at least the following regularity (see e.g. [9, § 1.4] for the basic properties of Bochner spaces)

$$w \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W), \tag{2.7}$$

$$\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{2.8}$$

$$\xi \in L^2(0,T;H), \tag{2.9}$$

$$\sigma \in L^\infty(0,T;H). \tag{2.10}$$

Given $\rho > 0$, the problem is to find a quadruplet $(w, \varphi, \xi, \sigma)$ satisfying (2.7)–(2.10) and

$$(w_t + l\varphi)_t - \kappa \triangle w_t - \tau \triangle w + \rho\sigma = f \qquad \text{a.e. in } Q, \tag{2.11}$$
$$\sigma \in \operatorname{Sign}(w_t + \alpha\varphi - \eta^*) \qquad \text{in } H, \text{ a.e. in } (0,T), \tag{2.12}$$
$$\varphi_t - \triangle\varphi + \xi + \pi(\varphi) = \gamma w_t \qquad \text{a.e. in } Q, \tag{2.13}$$
$$\xi \in \beta(\varphi) \qquad \text{a.e. in } Q, \tag{2.14}$$
$$w_t(0) = \vartheta_0, \qquad w(0) = w_0, \qquad \varphi(0) = \varphi_0 \qquad \text{a.e. in } \Omega. \tag{2.15}$$

We note that the Neumann boundary condition $\partial_n w = \partial_n \varphi = 0$ is incorporated in the definition of the space W.

We can now state the existence theorem.

Theorem 2.2 (Existence)**.** Assume (2.1)–(2.4), (2.5)–(2.6). Then there exist two constants C_1, $C_2 > 0$ such that for every $\rho > 0$ the problem (2.11)–(2.15) has at least a solution $(w, \varphi, \xi, \sigma)$ satisfying (2.7)–(2.10) and the following estimates hold

$$\begin{aligned} &\|w\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} + \|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \\ &+ \rho\|w_t + \alpha\varphi - \eta^*\|_{L^1(0,T;H)} + \|\widehat{\beta}(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \\ &+ \|\xi\|_{L^2(0,T;H)} + \|\sigma\|_{L^\infty(0,T;H)} \le C_1, \end{aligned} \tag{2.16}$$

$$\|w\|_{H^2(0,T;H)} \le C_2(1 + \rho^{1/2}). \tag{2.17}$$

The following result gives further regularity of the solutions under the hypothesis

$$\varphi_0 \in W \qquad \text{and} \qquad \beta^\circ(\varphi_0) \in H. \tag{2.18}$$

The regularity given by this theorem is necessary for proving the existence of sliding modes.

Theorem 2.3 (Further regularity)**.** Assume the hypotheses of Theorem 2.2 and the condition (2.18). Then every solution $(w, \varphi, \xi, \sigma)$ given by Theorem 2.2 satisfies

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.19}$$
$$\xi \in L^\infty(0,T;H), \tag{2.20}$$

and

$$\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\xi\|_{L^\infty(0,T;H)} \le C_3(1 + \rho^{1/2}), \tag{2.21}$$

where $C_3 > 0$ is a constant independent of ρ.

The next theorem closes the topic of the well-posedness of the problem. As consequence, we have that under the assumption $l = \alpha$ the solution is unique.

Theorem 2.4 (Continuous dependence)**.** Suppose (2.1)–(2.3), (2.5), $\rho > 0$, and $l = \alpha$.

Let $i = 1, 2$. We consider $(\vartheta_{0,i}, w_{0,i}, \varphi_{0,i}, f_i, w_i, \varphi_i, \xi_i, \sigma_i)$ where the functions $(\vartheta_{0,i}, w_{0,i}, \varphi_{0,i})$ are initial data satifying equation (2.6), f_i is a function satisfying (2.4), and $(w_i, \varphi_i, \xi_i, \sigma_i)$ is a solution of the problem given by Theorem 2.2 with $(\vartheta_0, w_0, \varphi_0) = (\vartheta_{0,i}, w_{0,i}, \varphi_{0,i})$ and $f = f_i$.

Then, there exists a constant C_4 independent of $\vartheta_{0,i}, w_{0,i}, \varphi_{0,i}, f_i$, and ρ such that

$$\begin{aligned} &\|w_1 - w_2\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\le C_4(\|\vartheta_{0,1} - \vartheta_{0,1}\|_H + \|w_{0,1} - w_{0,2}\|_V \\ &+ \|\varphi_{0,1} - \varphi_{0,2}\|_H + \|f_1 - f_2\|_{L^2(Q)}). \end{aligned} \tag{2.22}$$

Corollary 2.5 (Uniqueness)**.** Suppose that the hypotheses of Theorem 2.2 hold true and that $l = \alpha$. Then the solution is unique.

Finally we come to the most important result of this work: the theorem that guarantees that the solutions reach the sliding manifold in a finite time.

Theorem 2.6 (Sliding mode)**.** Assume (2.1)–(2.4), (2.5)–(2.6), (2.18), and $f \in L^\infty(0, T; H)$. Then there exist $\rho^* > 0$, such that the following condition is fulfilled.

For every $\rho > \rho^*$ and for every solution $(w, \varphi, \xi, \sigma)$ to the problem (2.11)–(2.15) there exist a time $T^* \in [0, T)$, such that

$$w_t(t) + \alpha\varphi(t) = \eta^*, \qquad \text{a.e. in } \Omega, \text{ for a.a. } t \in (T^*, T). \tag{2.23}$$

Remark 2.7. The statement of the theorem gives no estimates for ρ^* and T^*, but in the proof we will find certain bounds which we summarize here. Define C_5 and ψ_0 as

$$C_5 = \tau C_1 + (\kappa\alpha + |\alpha - l|)C_3 + \kappa\|\triangle\eta^*\|_H + \|f\|_{L^\infty(0,T;H)}, \tag{2.24}$$

$$\psi_0 = \|\vartheta_0 + \alpha\varphi_0 - \eta^*\|_H \tag{2.25}$$

where the constants C_1 and C_3 are given by Theorems 2.2 and 2.3, respectively. The quantity ψ_0 measures how far the initial state is from the sliding manifold. We will see that it is sufficient to choose

$$\rho^* = 2\left(\frac{\psi_0}{T} + C_5 + \frac{C_5^2}{2}\right) \tag{2.26}$$

to fulfill the condition described by the theorem. Moreover, for a given $\rho > \rho^*$, we will prove the following bound on T^*

$$T^* \leq \frac{2\psi_0}{\rho - 2C_5 - C_5^2} < T. \tag{2.27}$$

Equation (2.26) is consistent with the heuristic idea that, the greater the distance of the initial data from the sliding manifold is, the greater the sliding mode parameter ρ must be. On the contrary, with a small final time T, the solutions must reach the sliding manifold more quickly, thus ρ is forced to be large. Furthermore, although ψ_0 might be 0, i.e. the initial data lie in the sliding manifold, nothing ensures that ρ^* is 0, that is the solutions evolve in the sliding manifold.

Equation (2.27) estimates the time T^* when the evolution reaches the sliding manifold, i.e. the constraint starts to be fulfilled. In particular T^* is 0, provided that $\psi_0 = 0$.

Remark 2.8. The condition $f \in L^\infty(0,T;H)$ is not strictly necessary, because one may just assume that there exist a time $\widehat{T} \in [0,T)$ such that $f|_{[\widehat{T},T]} \in L^\infty(\widehat{T},T;H)$. One can make the system evolve in $[0,\widehat{T}]$ thanks to Theorem 2.2 and then apply Theorem 2.6 for the time interval $[\widehat{T},T]$. The theorem of the existence of sliding modes is stated with the stronger hypothesis $f \in L^\infty(0,T;H)$ only for simplicity and clearness.

We now see that the weaker hypothesis that $f|_{[\widehat{T},T]} \in L^\infty(\widehat{T},T;H)$ for a $\widehat{T} \in [0,T)$ is also necessary. Let $\eta : [0,T] \to H$ be a function defined as (later in this work we will define η in the same or a similar way)

$$\eta = w_t + \alpha\varphi - \eta^*.$$

Suppose the existence of a sliding mode, that is for $\rho > 0$, there is a time $T^* \in (0,T)$ such that η vanishes in (T^*,T). If η vanishes then also $\partial_t \eta$ and $\triangle\eta$ vanish. Thus, for $t > T^*$ we can rewrite equation (2.11) as

$$(l-\alpha)\varphi_t + \kappa\alpha\,\triangle\varphi - \kappa\,\triangle\eta^* - \tau\,\triangle w + \rho\sigma = f, \qquad \text{a.e. in } \Omega\times(T^*,T). \tag{2.28}$$

By Theorem 2.3 we have that all the summands in the left side of the above equation belong to $L^\infty(T^*,T;H)$ and so is the same for f.

2.1 Some tools

We recall for the reader's convenience a few useful facts that will be used throughout this work.

The first fact is a slightly modified version of the Young inequality. For $a, b, r \in \mathbb{R}$ and $a, b, r > 0$ it holds true that

$$ab \leq \frac{1}{2r}a^2 + \frac{r}{2}b^2. \tag{2.29}$$

We refer to the equation above as the Young inequality.

We will make use of the following Gronwall lemma [13, Th. 5.1, p. 498].

Lemma 2.9 (Gronwall)**.** Assume $u : (0,T) \to \mathbb{R}$ is a non-negative, bounded, measurable function and $a, b \in \mathbb{R}$, $a, b \geq 0$. If for all $t \in [0,T]$

$$u(t) \leq a + b\int_0^t u(s)\mathrm{d}s,$$

then for all $t \in (0,T)$

$$u(t) \leq a\exp(bt).$$

Finally we state a theorem by Stampacchia, which allows us to differentiate the composition of a Lipschitz-continuous function with a Sobolev function.

Theorem 2.10 (Stampacchia)**.** Assume $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous function and $u \in V$. Let A the set of all $x \in \Omega$ such that G' is defined in $u(x)$ and $B = \Omega \backslash A$. Then

$$\begin{aligned}
&G \circ u \in V, \\
&\nabla u = 0 \qquad \text{a.e. in } B, \\
&\nabla(G \circ u) = G'(u)\nabla u,
\end{aligned}$$

where the last equation has to be interpreted in the following sense: if $x \in A$ then $\nabla(G \circ u)(x) = G'(u(x))\nabla u(x)$, while $\nabla(G \circ u)(x) = 0$ if $x \in B$.

Chapter 3

Existence proof

The proof of the existence in Theorem 2.2 goes along the following this line. First of all, we will introduce the Yosida approximations of Sign and β. In the following section, we will make use of the Faedo–Galerkin method, in order to approximate the solutions. Then we will make certain a priori estimates that give uniform bounds of the approximate solutions. Finally we will take the limit of the approximate solutions and we will prove that the limit is actually a solution to the problem.

In this chapter, we denote with C a time-to-time-different, positive, large-enough constant independent of ρ and ε (the paramereter ε will be introduced in the following section).

3.1 Yosida approximations

In this section we recall a few facts regarding the theory of the Yosida approximation of maximal monotone operators and the Moreau-Yosida regularization of convex functions (see e.g. [4, Ch. 2] or [1, Ch. 2] for an introduction to Yosida approximation; see [11, Ch. 15] for Yosida regularization in metric spaces). After considering the abstract case, we will soon apply the results to the functions $\widehat{\beta}$ and $\|\cdot\|_H$.

We start with the definition of Moreau-Yosida regularization. Given a Hilbert space X (whose norm is denoted by $\|\cdot\|$), a proper, convex, l.s.c. function $\Phi : X \to [0, +\infty]$, and $\varepsilon > 0$, we define the Moreau-Yosida regularization Φ_ε as

$$\Phi_\varepsilon(v) = \inf_{w\in X} \left\{ \tfrac{1}{2\varepsilon}\|v - w\|^2 + \Phi(w) \right\}. \tag{3.1}$$

We incidentally notice that the infimum in the definition above is attained. The following proposition summarize the properties, which we will use later on, of the Moreau-Yosida regularization.

Proposition 3.1. Let $\Phi : X \to [0, +\infty]$ be a convex, proper, l.s.c. function. Then, the following conclusions hold

1) Φ_ε is convex and continuous;

2) $\Phi_\varepsilon(v) < +\infty$ and $\Phi_\varepsilon(v) \le \Phi(x)$ for all $v \in X$;

3) $\Phi_\varepsilon(v)$ converges monotonically to $\Phi(v)$ as $\varepsilon \to 0$;

4) $\liminf_{\varepsilon\to 0} \Phi_\varepsilon(v_\varepsilon) \ge \Phi(v)$, if v_ε is a sequence converging to v;

5) Φ_ε is Fréchet-differentiable, the differential $\partial\Phi_\varepsilon$ is ε^{-1}-Lipschitz-continuous, and

$$\|\partial\Phi_\varepsilon(x)\| \le \|y\| \qquad \forall x \in X, y \in \partial\Phi(x).$$

The differential $\partial\Phi_\varepsilon$ coincides with the Yosida approximation of the maximal monotone operator $\partial\Phi$.

At this point, we introduce $\widehat{\beta}_\varepsilon : \mathbb{R} \to \mathbb{R}$ and $\beta_\varepsilon = \partial\widehat{\beta}_\varepsilon : \mathbb{R} \to \mathbb{R}$, the Moreau-Yosida regularization of $\widehat{\beta}$ and Yosida approximation of β, respectively. It follows immediately from the previous proposition and the facts that $\widehat{\beta}(0) = 0$ and $\beta(0) \ni 0$, that

$$\beta_\varepsilon(0) = 0, \qquad \widehat{\beta}_\varepsilon(0) = 0, \tag{3.2}$$

$$|\beta_\varepsilon(r) - \beta_\varepsilon(s)| \le \frac{1}{\varepsilon}|r-s|, \qquad 0 \le \widehat{\beta}_\varepsilon(r) \le \frac{1}{2\varepsilon}r^2, \tag{3.3}$$

$$|\beta_\varepsilon(r)| \le |\beta^\circ(r)|, \qquad \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r), \tag{3.4}$$

for all $t, s \in \mathbb{R}$, where $\beta^\circ(r)$ denotes the element of $\beta(r)$ having minimum modulus.

In the same way, we introduce the Moreau-Yosida regularization $\|\cdot\|_{H,\varepsilon} : H \to \mathbb{R}$ and the Yosida approximation $\mathrm{Sign}_\varepsilon : H \to H$. It holds that

$$\|v\|_{H,\varepsilon} := \min_{w\in H}\{\tfrac{1}{2\varepsilon}\|v-w\|_H^2 + \|w\|_H\} = \begin{cases} \|v\|_H - \dfrac{\varepsilon}{2} & \text{if } \|v\|_H \ge \varepsilon, \\ \dfrac{\|v\|_H^2}{2\varepsilon} & \text{if } \|v\|_H \le \varepsilon. \end{cases} \tag{3.5}$$

Indeed, if we differentiate the convex function $w \mapsto \frac{1}{2\varepsilon}\|v-w\|_H^2 + \|w\|_H$, we obtain

$$w + \varepsilon \,\mathrm{Sign}\, w \ni v,$$

yielding

$$w = \begin{cases} (1 - \frac{\varepsilon}{\|v\|_H})v & \text{if } \|v\|_H \ge \varepsilon, \\ 0 & \text{if } \|v\|_H \le \varepsilon, \end{cases}$$

thus we can substitute w in the minimum of equation (3.5). We calculate the Yosida approximation of Sign, by differentiating (3.5), obtaining

$$\mathrm{Sign}_\varepsilon(v) = \frac{v}{\max\{\varepsilon, \|v\|_H\}} = \begin{cases} \dfrac{v}{\|v\|_H} & \text{if } \|v\|_H \geq \varepsilon, \\ \dfrac{v}{\varepsilon} & \text{if } \|v\|_H \leq \varepsilon, \end{cases} \tag{3.6}$$

which imply

$$(\mathrm{Sign}_\varepsilon(v), v) \geq \|v\|_{H,\varepsilon}. \tag{3.7}$$

Finally, we present the following identity, which will be useful later,

$$\|v\|_{H,\varepsilon} = \min\{s/\varepsilon, 1\}\mathrm{d}s. \tag{3.8}$$

Indeed, we have that

$$\begin{aligned} \|v\|_{H,\varepsilon} &= \int_0^1 (\mathrm{Sign}_\varepsilon(rv), v)\mathrm{d}r \\ &= \int_0^{\|v\|_H} \left(\mathrm{Sign}_\varepsilon\left(\frac{sv}{\|v\|_H}\right), v\right) \frac{1}{\|v\|_H}\mathrm{d}s \\ &= \int_0^{\|v\|_H} \left(\frac{sv/\|v\|_H}{\max\{\varepsilon, s\}}, \frac{v}{\|v\|_H}\right) \mathrm{d}s = \int_0^{\|v\|_H} \min\{s/\varepsilon, 1\}\mathrm{d}s. \end{aligned}$$

3.2 Faedo–Galerkin approximation

In order to use the Faedo–Galerkin method, we need to introduce a few notations. We take $\{v_i\}_{i=1}^{+\infty}$ a complete orthogonal set of V given by the eigenfunctions of the Laplace operator coupled with Neumann conditions, i.e.

$$-\triangle v_i = \lambda_i v_i \text{ on } \Omega, \qquad \partial_n v_i = 0 \text{ on } \Gamma,$$

where $\lambda_i \leq \lambda_{i+1}$, $i \in \mathbb{N}$, are the eigenvalues of the Laplace operator. We define $V_n := \mathrm{span}\{v_1, \dots, v_n\}$ and let $P_n : V \to V$ be the orthogonal projector on V_n. We know that $\cup_{i=1}^{+\infty} V_n$ is dense in V.

It is still true that $\{v_i\}_{i=1}^{+\infty}$ is a complete orthogonal set for H and W. Moreover, the operator P_n can be extended or restricted to H and W respectively and the extension and the restriction are still orthogonal projectors in the spaces H and W. We recall that if $v \in X$ then

$$P_n(v) \to v \text{ strongly in } X \qquad \text{and} \qquad \|P_n(v)\|_X \leq \|v\|_X, \tag{3.9}$$

where X can be either H, V or W. We now project the initial data and the target function η^*

$$\vartheta_{0,n} := P_n\vartheta_0, \qquad w_{0,n} := P_n w_0, \qquad \varphi_{0,n} := P_n\varphi_0, \qquad \eta_n^* := P_n\eta^*.$$

Finally, using standard density results, we take $f_n \in C([0,T];H)$, such that f_n converges strongly to f in $L^2(0,T;H)$.

The new problem is now to find two functions $w_n \in C^2([0,T];V_n)$ and $\varphi_n \in C^1([0,T];V_n)$, such that

$$\begin{aligned}&(\partial_t^2 w_n + l\partial_t\varphi_n - \kappa\,\triangle\partial_t w_n - \tau\,\triangle w_n\\&+ \rho\,\mathrm{Sign}_\varepsilon(\partial_t w_n + \alpha\varphi_n - \eta_n^*), v) = (f_n, v), \qquad \forall v \in V_n,\ \text{in } [0,T],\end{aligned} \tag{3.10}$$

$$(\partial_t\varphi_n - \triangle\varphi_n + \beta_\varepsilon(\varphi_n) + \pi(\varphi_n), v) = \gamma(\partial_t w_n, v), \qquad \forall v \in V_n,\ \text{in } [0,T], \tag{3.11}$$

$$\partial_t w_n(0) = \vartheta_{0,n}, \qquad w_n(0) = w_{0,n}, \qquad \varphi_n(0) = \varphi_{0,n}. \tag{3.12}$$

This is a non-linear system of ordinary differential equations of the second and first order in the variables w_n and φ_n respectively. The non-linearity is only given by $\mathrm{Sign}_\varepsilon$, β_ε, and π, which are all Lipschitz-continuous functions. Hence, by Cauchy-Lipschitz theorem, there exists a unique solution (w_n, φ_n) defined on $[0,T]$.

3.3 A priori estimates

3.3.1 First a priori estimate

We test equation (3.10) and equation (3.11) by taking $v = \partial_t w_n + \alpha\varphi_n - \eta_n^*$ and $v = \partial_t\varphi_n$. We sum with $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_n\|_H^2 - (\varphi_n, \partial_t\varphi_n) = 0$ obtaining

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t w_n\|_H^2 + (\partial_t^2 w_n, \alpha\varphi_n - \eta_n^*) + l(\partial_t\varphi_n, \partial_t w_n + \alpha\varphi_n - \eta_n^*)\\&+ \kappa\|\partial_t\nabla w_n\|_H^2 + \kappa(\partial_t\nabla w_n, \nabla(\alpha\varphi_n - \eta_n^*))\\&+ \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w_n\|_H^2 + \tau(\nabla w_n, \nabla(\alpha\varphi_n - \eta_n^*))\\&+ \rho(\mathrm{Sign}_\varepsilon(\partial_t w_n + \alpha\varphi_n - \eta_n^*), \partial_t w_n + \alpha\varphi_n - \eta_n^*)\\&+ \|\partial_t\varphi_n\|_H^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi_n\|_H^2 + \|\nabla\varphi_n\|_H^2)\\&+ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \widehat{\beta}_\varepsilon(\varphi_n) + ((\pi(\varphi_n) - \varphi_n), \partial_t\varphi_n)\\&= (f_n(t), \partial_t w_n + \alpha\varphi_n - \eta_n^*) + \gamma(\partial w_n, \partial_t\varphi_n).\end{aligned}$$

We integrate between 0 and t, and, recalling that $(\mathrm{Sign}_\varepsilon(v), v) \geq \|v\|_{H,\varepsilon}$, we get

$$
\begin{aligned}
&\frac{1}{2}\|\partial_t w_n(t)\|_H^2 + \kappa \int_0^t \|\partial_t \nabla w_n\|_H^2 + \frac{\tau}{2}\|\nabla w_n(t)\|_H^2 \\
&+ \rho \int_0^t \|\partial_t w_n + \alpha\varphi_n + \eta_n^*\|_{H,\varepsilon} + \int_0^t \|\partial_t \varphi_n\|_H^2 \\
&+ \frac{1}{2}\|\varphi_n(t)\|_V^2 + \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t)) \\
&\leq \frac{1}{2}\|\vartheta_{0,n}\|_H^2 + \frac{\tau}{2}\|\nabla w_{0,n}\|_H^2 + \frac{1}{2}\|\varphi_{0,n}\|_V^2 + \|\widehat{\beta}_\varepsilon(\varphi_{0,n})\|_{L^1(\Omega)} \\
&+ \int_0^t (\partial_t w_n, \alpha\partial_t\varphi_n) + (\vartheta_{0,n}, \alpha\varphi_{0,n} - \eta_n^*) - (\partial_t w_n(t), \alpha\varphi(t) - \eta_n^*) \\
&- l \int_0^t (\partial_t\varphi_n, \partial_t w_n + \alpha\varphi_n - \eta_n^*) - \kappa \int_0^t (\partial_t \nabla w_n, \nabla(\alpha\varphi_n - \eta_n^*)) \\
&- \tau \int_0^t (\nabla w_n, \nabla(\alpha\varphi_n - \eta_n^*)) - \int_0^t (\pi(\varphi_n) - \varphi_n, \partial_t\varphi_n) \\
&+ \int_0^t (f_n, \partial_t w_n + \alpha\varphi_n - \eta_n^*) + \int_0^t (\partial_t w_n, \partial_t \varphi_n).
\end{aligned}
\tag{3.13}
$$

We need now to control the summands of the left side of (3.13). By (3.9) we have that

$$\|\vartheta_{0,n}\|_H^2 \leq \|\vartheta_0\|_H^2 \leq C,$$

and in the same way we can control $\|\nabla w_{0,n}\|_H$ and $\|\varphi_{0,n}\|_V$. For the last initial datum we note

$$\int_\Omega \widehat{\beta}_\varepsilon(\varphi_n) \leq \int_\Omega \frac{1}{2\varepsilon}|\varphi_{0,n}|^2 \leq \frac{1}{2\varepsilon}\|\varphi_0\|_H^2. \tag{3.14}$$

Using the Young Inequality (2.29) we find

$$\int_0^t (\partial_t w_n, \alpha\partial_t\varphi_n) \leq \frac{\alpha^2}{2}\int_0^t \|\partial_t w_n\|_H^2 + \frac{1}{2}\int_0^t \|\partial_t\varphi_n\|_H^2.$$

The next step is easier

$$(\vartheta_{0,n}, \alpha\varphi_{0,n} - \eta_n^*) \leq \|\vartheta_0\|_H^2 + \frac{\alpha^2}{2}\|\varphi_0\|_H^2 + \frac{1}{2}\|\eta^*\|_H^2 \leq C.$$

Again, owing to Young inequality, we infer

$$
\begin{aligned}
-(\partial_t w_n(t), \alpha\varphi_n(t) - \eta_n^*) &\leq \frac{1}{4}\|\partial_t w_n(t)\|_H^2 + \|\alpha\varphi_n(t) - \eta_n^*\|_H^2 \\
&\leq \frac{1}{4}\|\partial_t w_n(t)\|_H^2 + 2\alpha^2\|\varphi_n(t)\|_H^2 + C.
\end{aligned}
$$

Since

$$\begin{aligned}\|\varphi_n(t)\|_H^2 &= \|\varphi_{0,n}\|_H^2 + 2\int_0^t (\varphi_n, \partial_t\varphi_n) \\ &\leq \|\varphi_0\|_H^2 + 8\alpha^2 \int_0^t \|\varphi_n\|_H^2 + \frac{1}{8\alpha^2}\int_0^t \|\partial_t\varphi_n\|_H^2,\end{aligned}$$

we find

$$-(\partial_t w_n(t), \alpha\varphi_n(t) - \eta_n^*) \leq \frac{1}{4}\|\partial_t w_n(t)\|_H^2 + \frac{1}{4}\int_0^t \|\partial_t \varphi\|_H^2 + C\left(1 + \int_0^t \|\varphi_n\|_H^2\right).$$

Using the same technique we deduce

$$\begin{aligned}&- l\int_0^t (\partial_t\varphi_n, \partial_t w_n + \alpha\varphi_n - \eta_n^*) \\ &\leq \frac{1}{8}\int_0^t \|\partial_t\varphi_n\|_H^2 + C\left(1 + \int_0^t \|\partial_t w_n\|_H^2 + \int_0^t \|\varphi_n\|_H^2\right),\end{aligned}$$

$$-\kappa\int_0^t (\partial_t \nabla w_n, \nabla(\alpha\varphi_n - \eta_n^*) \leq \frac{\kappa}{2}\int_0^t \|\partial_t\nabla w_n\|_H^2 + \alpha^2\int_0^t \|\varphi_n\|_H^2 + C,$$

$$-\tau\int_0^t (\nabla w_n, \nabla(\alpha\varphi_n + \eta_n^*) \leq \frac{\tau}{2}\int_0^t \|\nabla w_n\|_H^2 + \tau\alpha^2\int_0^t \|\nabla\varphi_n\|_H^2 + C.$$

Then, recalling that π is Lipschitz-continuous, we have

$$\begin{aligned}-\int_0^t (\pi(\varphi_n) - \varphi_n, \partial_t\varphi_n) &\leq 4\int_0^t \|\pi(\varphi_n) - \varphi_n\|_H^2 + \frac{1}{16}\int_0^t \|\partial_t\varphi_n\|_H^2 \\ &\leq C\left(1 + \int_0^t \|\varphi_n\|_H^2\right) + \frac{1}{16}\int_0^t \|\partial_t\varphi_n\|_H^2,\end{aligned}$$

$$\int_0^t (f_n, \partial_t w_n + \alpha\varphi_n - \eta_n^*) \leq C\left(1 + \int_0^t \|\partial_t w_n\|_H^2 + \int_0^t \|\varphi_n\|_H^2\right),$$

$$\int_0^t (\partial_t w_n, \partial_t\varphi_n) \leq 8\int_0^t \|\partial_t w_n\|_H^2 + \frac{1}{32}\int_0^t \|\partial_t\varphi_n\|_H^2.$$

We put everything together, obtaining

$$\begin{aligned}&\frac{1}{32}\|\partial_t w_n(t)\|_H^2 + \frac{\kappa}{2}\int_0^t \|\partial_t\nabla w_n\|_H^2 + \frac{\tau}{2}\|\nabla w_n(t)\|_H^2 \\ &+ \rho\int_0^t \|\partial_t w_n + \alpha\varphi_n + \eta_n^*\|_{H,\varepsilon} + \frac{1}{4}\int_0^t \|\partial_t\varphi_n\|_H^2 + \frac{1}{2}\|\varphi_n(t)\|_V^2 \\ &+ \int_\Omega \widehat{\beta}_\varepsilon(\varphi_n(t)) \leq C\left(1 + \varepsilon^{-1} + \int_0^t \|\partial_t w_n\|_H^2 + \int_0^t \|\varphi_n\|_V^2 + \int_0^t \|\nabla w_n\|_H^2.\right)\end{aligned}$$

We use now the Gronwall Lemma deducing

$$\begin{aligned} &\|w_n\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|\varphi_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\ &+ \rho \int_0^T \|\partial_t w_n + \alpha\varphi_n + \eta_n^*\|_{H,\varepsilon} + \|\widehat{\beta}_\varepsilon(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} \leq C(1+\varepsilon^{-1}). \end{aligned} \tag{3.15}$$

Remark 3.2. Unfortunately, we are not able to provide an estimate independent of ε, preventing the possibility of taking the limit as $\varepsilon \to 0$. For the moment, we do not worry about this trouble since our first aim is to take the limit as $n \to \infty$.

We stress that the only dependence on ε in (3.15) arises in (3.14). We anticipate that the other estimates have a dependence on ε because we will use this estimate to prove them. Hence, when we will be able to fine-tune equation (3.14) removing the dependence on ε, all estimates will work perfectly, being independent of ε.

Finally, we point out that the term $C(1+\varepsilon^{-1})$ could be slightly refined in $C(1+\varepsilon^{-1/2})$. We will be sloppy in carrying out the dependence on ε because, as we have just said, we will remove this dependence, and, at this stage, we only want estimates independent of n.

3.3.2 Second a priori estimate

We define $g_1 : [0,T] \to H$ as $g_1(t) = \gamma\partial_t w_n(t) - \partial_t\varphi_n(t) - \pi(\varphi_n(t))$. Due to the first a priori estimate and the Lipschitz-continuity of π we have that

$$\|g_1\|_{L^2(0,T;H)} \leq C(1+\varepsilon^{-1}).$$

We rewrite (3.11) as

$$-(\triangle\varphi_n(t), v) + (\beta_\varepsilon(\varphi_n(t)), v) = (g_1(t), v),$$

and we test with $v = -\triangle\varphi_n(t)$

$$\|\triangle\varphi_n(t)\|_H^2 - (\beta_\varepsilon(\varphi_n(t)), \triangle\varphi_n(t)) = -(g_1, \triangle\varphi_n) \leq \|g_1(t)\|_H \|\triangle\varphi_n(t)\|_H.$$

The second term is positive, because of

$$-(\beta_\varepsilon(\varphi_n), \triangle\varphi_n) = -\int_\Omega \beta_\varepsilon(\varphi_n)\,\triangle\varphi_n = \int_\Omega \beta_\varepsilon'(\varphi_n)|\nabla\varphi_n|^2 \geq 0,$$

then yielding $\|\triangle\varphi_n(t)\|_H \leq \|g_1(t)\|_H$. Since $P_n(\beta_\varepsilon(\varphi_n)) = P_n(g_1) + \triangle\varphi_n$, owing to elliptic regularity, we conclude that

$$\|\varphi_n\|_{L^2(0,T;W)} + \|P_n(\beta_\varepsilon(\varphi_n))\|_{L^2(0,T;H)} \leq C(1+\varepsilon^{-1}). \tag{3.16}$$

3.3.3 Third a priori estimate

We define $\eta, g_2 : [0,T] \to V_n$ as

$$\eta(t) := \partial_t w_n(t) + \alpha\varphi_n(t) - \eta_n^*$$

$$\begin{aligned} g_2(t) := {} & (\alpha - l)\partial_t\varphi_n(t) - \alpha\kappa \triangle\varphi_n(t) + \kappa \triangle\eta_n^* \\ & + \tau \triangle \left(w_{0,n} + \alpha \int_0^t \varphi_n(s)\mathrm{d}s + t\eta_n^* \right) + f_n(t). \end{aligned}$$

Thanks to equations (3.15) and (3.16) we have that

$$\|g_2\|_{L^2(0,T;H)} + \|\eta\|_{L^2(0,T;H)} \leq C(1 + \varepsilon^{-1}).$$

Moreover, equation (2.6) implies

$$\|\eta(0)\|_V = \|\vartheta_{0,n} + \alpha\varphi_{0,n} + \eta_n^*\|_V \leq C.$$

We rewrite equation (3.10) as

$$\left(\partial_t\eta - \kappa \triangle\eta - \tau \int_0^t \triangle\eta(s)\mathrm{d}s + \rho\,\mathrm{Sign}_\varepsilon(\eta), v \right) = (g_2, v).$$

In view of equation (3.5), it is clear that

$$\begin{aligned} \left(\triangle\eta(t), \int_0^t \triangle\eta(s)\mathrm{d}s \right) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\| \int_0^t \triangle\eta(s)\mathrm{d}s \right\|_H^2, \\ -\int_\Omega \mathrm{Sign}_\varepsilon(\eta) \triangle\eta &= \int_\Omega \nabla\,\mathrm{Sign}_\varepsilon(\eta) \cdot \nabla\eta \geq 0, \\ -\int_\Omega \partial_t\eta \triangle\eta &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla\eta|^2. \end{aligned}$$

Thus, we test (3.3.3) with $v = -\triangle\eta(t)$ and we integrate over time finding

$$\begin{aligned} \frac{1}{2}\|\nabla\eta(t)\|_H^2 + \kappa \int_0^t \|\triangle\eta\|_H^2 + \frac{\tau}{2}\left\| \int_0^t \triangle\eta\mathrm{d}s \right\|_H^2 & \\ \leq \frac{1}{2}\|\nabla\eta(0)\|_H^2 - \int_0^t (g_2, \triangle\eta) & \\ \leq C + \frac{\kappa}{2}\int_0^t \|\triangle\eta\|_H^2 + \frac{1}{2\kappa}\int_0^t \|g_2\|_H^2. & \end{aligned}$$

Hence, we infer

$$\|\nabla\eta\|_{L^\infty(0,T;H)} + \|\triangle\eta\|_{L^2(0,T;H)} \leq C(1 + \varepsilon^{-1}),$$

which, together with elliptic regularity, implies $\|\eta\|_{L^2(0,T;W)} \leq C(1+\varepsilon^{-1})$, thus

$$\|\partial_t w_n\|_{L^\infty(0,T;V)} + \|\partial_t w_n\|_{L^2(0,T;W)} \leq C(1+\varepsilon^{-1}).$$

Finally, as $\|w_n\|_{L^\infty(0,T;H)} \leq C(1+\varepsilon^{-1})$ and $w_{0,n} \in W$, we conclude that

$$\|w_n\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \leq C(1+\varepsilon^{-1}). \tag{3.17}$$

3.3.4 Fourth a priori estimate

We define $g_3 : [0,T] \to V_n$ as

$$g_3(t) = g_2(t) + \kappa \triangle\eta(t) + \tau \int_0^t \triangle\eta(s)\mathrm{d}s.$$

Again, it holds true that $\|g_3\|_{L^2(0,T;H)} \leq C(1+\varepsilon^{-1})$ by comparison. We rewrite equation (3.3.3) as

$$(\eta_t + \rho\,\mathrm{Sign}_\varepsilon(\eta), v) = (g_3, v). \tag{3.18}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{H,\varepsilon} = (\mathrm{Sign}_\varepsilon(\eta), \partial_t\eta)_H$, we can test (3.18) with $v = \partial_t\eta$ obtaining

$$\begin{aligned}\int_0^t \|\partial_t\eta\|_H^2 + \rho\|\eta(t)\|_H &= \rho\|\eta(0)\|_H + \int_0^t (g_3(s), \partial_t\eta(s))\mathrm{d}s \\ &\leq \rho C + \frac{1}{2}\int_0^t \|\partial_t\eta\|_H^2 + \frac{1}{2}\int_0^t \|g_3\|_H^2.\end{aligned}$$

Thus, $\|\partial_t\eta\|^2_{L^2(0,T;H)} \leq C(1+\rho+\varepsilon^{-1})$ and then, by comparison, we obtain

$$\|w_n\|_{H^2(0,T;H)} \leq C(1+\rho^{1/2}+\varepsilon^{-1}). \tag{3.19}$$

3.4 Passage to the limit in the Faedo–Galerkin scheme

Making use of standard weak or weak* compactness results, possibly taking a subsequence, we have that (w_n, φ_n) converges in the following topologies

$$w_n \to w_\varepsilon \qquad \text{weakly in } H^2(0,T;H) \cap H^1(0,T;W), \tag{3.20}$$

$$\varphi_n \to \varphi_\varepsilon \qquad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;W), \tag{3.21}$$

$$w_n \to w_\varepsilon \qquad \text{weakly* in } W^{1,\infty}(0,T;V), \tag{3.22}$$

$$\varphi_n \to \varphi_\varepsilon \qquad \text{weakly* in } L^\infty(0,T;V), \tag{3.23}$$

for a suitable pair $(w_\varepsilon, \varphi_\varepsilon)$. This implies, together with the generalized Ascoli theorem and the Aubin-Lions theorem [27, Sec. 8, Cor. 4], the following strong convergences

$$w_n \to w_\varepsilon \qquad \text{in } H^1(0,T;V) \cap C^1([0,T];H), \tag{3.24}$$
$$\varphi_n \to \varphi_\varepsilon \qquad \text{in } C^0([0,T];H) \cap L^2(0,T;V). \tag{3.25}$$

Hence, we have the following convergences in $C^0([0,T];H)$

$$\begin{aligned} \operatorname{Sign}_\varepsilon(\partial_t w_n + \alpha\varphi_n - \eta_n^*) &\to \operatorname{Sign}_\varepsilon(\partial_t w_\varepsilon + \alpha\varphi_\varepsilon - \eta^*), \\ \pi(\varphi_n) &\to \pi(\varphi_\varepsilon), \\ \beta_\varepsilon(\varphi_n) &\to \beta_\varepsilon(\varphi_\varepsilon). \end{aligned}$$

We note that $\eta_n^* \to \eta^*$ and that the initial conditions hold true

$$\partial_t w_\varepsilon(0) = \vartheta_0, \qquad w_\varepsilon(0) = w_0, \qquad \varphi_\varepsilon(0) = \varphi_0.$$

Indeed, the property (3.9) implies the strong convergence of the initial data and the target function η^*. We take $n, h \in \mathbb{N}$ with $n > h$ and $v \in V_h \subset V_n$. Since all the involved terms converge, we take the limit as $n \to +\infty$ in equations (3.10) and (3.11), obtaining

$$\begin{aligned} &(\partial_t^2 w_\varepsilon + l\partial_t\varphi_\varepsilon - \kappa \triangle \partial_t w_\varepsilon - \tau \triangle w_\varepsilon \\ &\quad + \rho \operatorname{Sign}_\varepsilon(\partial_t w_\varepsilon + \alpha\varphi_\varepsilon - \eta^*), v) = (f, v) \qquad \forall v \in V_h, \text{ a.e. in } (0,T), \\ &(\partial_t\varphi_\varepsilon - \triangle\varphi_\varepsilon + \beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon), v) = \gamma(\partial_t w_\varepsilon, v) \qquad \forall v \in V_h, \text{ a.e. in } (0,T). \end{aligned}$$

As h is arbitrary, the above equations hold for all $v \in \cup_{h=1}^{+\infty} V_h$. By density of $\cup_{h=1}^{+\infty} V_h$ in H, we find

$$\begin{aligned} &(\partial_t^2 w_\varepsilon + l\partial_t\varphi_\varepsilon - \kappa \triangle \partial_t w_\varepsilon - \tau \triangle w_\varepsilon \\ &\quad + \rho \operatorname{Sign}_\varepsilon(\partial_t w_\varepsilon + \alpha\varphi_\varepsilon - \eta^*), v) = (f, v) \qquad \forall v \in H, \text{ a.e. in } (0,T), \end{aligned} \tag{3.26}$$

$$(\partial_t\varphi_\varepsilon - \triangle\varphi_\varepsilon + \beta_\varepsilon(\varphi_\varepsilon) + \pi(\varphi_\varepsilon), v) = \gamma(\partial_t w_\varepsilon, v) \qquad \forall v \in H, \text{ a.e. in } (0,T). \tag{3.27}$$

3.5 Passage to the limit as $\varepsilon \to 0$

Let $\xi_\varepsilon := \beta_\varepsilon(\varphi_\varepsilon)$ and $\sigma_\varepsilon := \operatorname{Sign}_\varepsilon(\partial_t w_\varepsilon + \alpha\varphi_\varepsilon - \eta^*)$. We now review the a priori estimates in order to remove the dependence on ε. All calculations are still working if we substitute $(w_n, \varphi_n, \beta_\varepsilon(\varphi_n), \operatorname{Sign}_\varepsilon(\partial_t w_n + \alpha\varphi_n - \eta_n^*))$

with $(w_\varepsilon, \varphi_\varepsilon, \xi_\varepsilon, \sigma_\varepsilon)$. By Remark 3.2, the dependence on ε is only given by equation (3.14). We observe that, owing to (3.4),

$$\|\widehat{\beta}_\varepsilon(\varphi_0)\|_{L^1(\Omega)} \leq \|\widehat{\beta}(\varphi_0)\|_{L^1(\Omega)},$$

and we had just made the first a priori estimate independent of ε:

$$\begin{aligned} &\|w_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|\varphi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\ &+ \rho \int_0^T \|\partial_t w_\varepsilon + \alpha\varphi_\varepsilon + \eta^*\|_{H,\varepsilon} + \|\widehat{\beta}_\varepsilon(\varphi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \end{aligned} \tag{3.28}$$

Having removed the dependence on ε in the first estimate, all other estimates can be replicated obtaining

$$\|\varphi_\varepsilon\|_{L^2(0,T;W)} + \|\xi_\varepsilon\|_{L^2(0,T;H)} \leq C, \tag{3.29}$$
$$\|w_\varepsilon\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} \leq C, \tag{3.30}$$
$$\|w_\varepsilon\|_{H^2(0,T;H)} \leq C(1+\rho^{1/2}). \tag{3.31}$$

Moreover, because of the definition of the Sign operator, $\sigma_\varepsilon(t)$ is bounded, uniformly with respect to t and ε, i.e.,

$$\|\sigma_\varepsilon\|_{L^\infty(0,T;H)} \leq 1. \tag{3.32}$$

We are now able to take the limit as $\varepsilon \to 0$ using the same compactness argument as before. There exists a quadruplet $(w, \varphi, \xi, \sigma)$ such that (a subsequence of) $(w_\varepsilon, \varphi_\varepsilon, \xi_\varepsilon, \sigma_\varepsilon)$ converges to $(w, \varphi, \xi, \sigma)$ in the same topologies as before. More precisely for ξ_ε and σ_ε we have that

$$\xi_\varepsilon \to \xi \qquad \text{weakly in } L^2(0,T;H), \tag{3.33}$$
$$\sigma_\varepsilon \to \sigma \qquad \text{weakly in } L^2(0,T;H). \tag{3.34}$$

We take the limit in equation (3.26) and (3.27) obtaining (2.11) and (2.13), respectively. Since φ_ε and $\partial_t w_\varepsilon$ converge strongly in $L^2(0,T;H) = L^2(Q)$ and ξ_ε and σ_ε converge weakly, we deduce

$$\lim_{\varepsilon\to 0} \int_Q \xi_\epsilon\, \varphi_\epsilon = \int_Q \xi\, \varphi,$$
$$\lim_{\varepsilon\to 0} \int_Q \sigma_\epsilon\, (\partial_t w_\epsilon + \alpha\varphi_\epsilon - \eta^*) = \int_Q \sigma\, (\partial_t w + \alpha\varphi - \eta^*).$$

Hence, by [1, Prop. 2.2, p. 38] we have that

$$\xi \in \beta(\varphi) \qquad \text{and} \qquad \sigma \in \operatorname{Sign}(\partial_t w + \alpha\varphi - \eta^*),$$

almost everywhere, and the proof of the existence of the solutions is complete.

To conclude the proof of Theorem 2.2 we need to prove (2.16) and (2.17). Owing to the lower semi-continuity of the norms, the Fatou lemma, and part 4 of Proposition 3.1, we can take the inferior limit as $\varepsilon \to 0$ in (3.28)–(3.32) deducing

$$\begin{aligned} &\|w\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} + \|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \\ &+ \rho\|w_t + \alpha\varphi - \eta^*\|_{L^1(0,T;H)} + \|\widehat{\beta}(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \\ &+ \|\xi\|_{L^2(0,T;H)} + \|\sigma\|_{L^\infty(0,T;H)} \leq C, \end{aligned} \tag{3.35}$$

$$\|w\|_{H^2(0,T;H)} \leq C(1+\rho^{1/2}). \tag{3.36}$$

Chapter 4

Further regularity

We now prove Theorem 2.3. For the sake of clarity, the proof can be divided in two parts. In the former part we use the notations of the Faedo–Galerkin scheme to prove that the limit function φ_ε is more regular (in particular $\varphi_\varepsilon \in H^1(0,T;V)$). In the latter part, with the conventions of Section 3.5, the desired estimate (2.21) is shown for the approximate solution φ_ε. Using the usual compactness argument and the lower semi-continuity of the norm, the estimate (2.21) will follow automatically. In this proof, C still denotes a positive constant independent of ρ.

We consider equation (3.11) and note that: 1) the functions $\triangle\varphi_n$ and $\partial_t w_n$ are derivable and their derivatives are $\triangle\partial_t\varphi_n$ and $\partial_t^2 w_n$, respectively; 2) for all $v \in V_n$, the functions

$$t \mapsto (\beta_\varepsilon(\varphi_n(t)), v) \qquad \text{and} \qquad t \mapsto (\pi(\varphi_n(t)), v)$$

are Lipschitz-continuous, thus derivable a.e. in $(0,T)$ with derivative

$$(\beta_\varepsilon'(\varphi_n(t))\partial_t\varphi_n(t), v) \qquad \text{and} \qquad (\pi'(\varphi_n(t))\partial_t\varphi_n(t), v),$$

respectively. Thus $\partial_t\varphi_n$ is Lipschitz-continuous by comparison and we can derive (in weak sense) equation (3.11) obtaining

$$(\partial_t^2\varphi_n - \triangle\partial_t\varphi_n + \beta_\varepsilon'(\varphi_n)\partial_t\varphi_n, v) = (g_4, v), \qquad \forall v \in V_n,\ \text{a.e. in } (0,T), \quad (4.1)$$

where

$$g_4 := -\pi'(\varphi_\varepsilon)\partial_t\varphi_n + \gamma\partial_t^2 w_n.$$

Clearly $\|g_4\|_{L^2(0,T;H)} \leq C(1+\varepsilon^{-1}+\rho^{1/2})$, as π' is bounded. We take $v = \partial_t\varphi_\varepsilon$

in equation (4.1) and we integrate over $(0,t)$ obtaining

$$\begin{aligned}
&\frac{1}{2}\|\partial_t\varphi_n(t)\|_H^2 + \int_0^t \|\nabla\partial_t\varphi_n\|_H^2 + \int_{Q_t} \beta_\varepsilon'(\varphi_n)|\partial_t\varphi_n|^2 \\
&\le \frac{1}{2}\|\partial_t\varphi_n(0)\|_H^2 + \int_0^t (g_4, \partial_t\varphi_n) \\
&\le \frac{1}{2}\|\partial_t\varphi_n(0)\|_H^2 + \frac{1}{2}\int_0^t \|g_4\|_H^2 + \frac{1}{2}\int_0^t \|\partial_t\varphi_n\|_H^2 \\
&\le \frac{1}{2}\|\partial_t\varphi_n(0)\|_H^2 + \frac{1}{2}\int_0^t \|\partial_t\varphi_n\|_H^2 + C(1+\varepsilon^{-2}+\rho).
\end{aligned}$$

Since β_ε is monotone, we have that $\beta_\varepsilon' \ge 0$ implies $\int_{Q_t} \beta_\varepsilon'(\varphi_n)|\partial_t\varphi_n|^2 \ge 0$. At this point, we want to use the Gronwall lemma to control $\|\partial_t\varphi_n\|_{L^2(0,T;H)}$. The most delicate part is to find a bound on $\|\partial_t\varphi_n(0)\|_H$. Using again equation (3.11) we compute

$$\begin{aligned}
\|\partial_t\varphi_n(0)\|_H &= \|\gamma\vartheta_{0,n} + \triangle\varphi_{0,n} - P_n(\beta_\varepsilon(\varphi_{0,n})) - P_n(\pi(\varphi_{0,n}))\|_H \\
&\le \gamma\|\vartheta_{0,n}\|_H + \|\triangle\varphi_{0,n}\|_H + \|P_n(\pi(\varphi_{0,n}))\|_H + \|P_n(\beta_\varepsilon(\varphi_{0,n}))\|_H \\
&\le \gamma\|\vartheta_0\|_H + \|\triangle\varphi_0\|_H + C\|\varphi_0\|_H + \varepsilon^{-1}\|\varphi_0\|_H \le C(1+\varepsilon^{-1}).
\end{aligned} \tag{4.2}$$

where the fact that β_ε is ε^{-1}-Lipschitz-continous and the hypothesis $\varphi_0 \in W$ have been taken into account. We incidentally note that we have not yet used the hypothesis $\beta^\circ(\varphi_0) \in H$. Owing to the Gronwall lemma, we obtain

$$\|\varphi_n\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le C(1+\varepsilon^{-1}+\rho^{1/2}). \tag{4.3}$$

In view of equations (3.21) and (3.23), we additionally have that

$$\varphi_n \to \varphi_\varepsilon \qquad \text{weakly* in } W^{1,\infty}(0,T;H)\cap H^1(0,T;V). \tag{4.4}$$

This proves that φ_ε belongs to $W^{1,\infty}(0,T;H)\cap H^1(0,T;V)$.

In this second part we refine our argument, removing the dependence on ε in the estimate (4.3). Like the former part of this proof, we want to derive equation (3.27). Since nothing ensures the weak-derivability of $\triangle\varphi_\varepsilon$, we take $v \in V$ and we reorganize (3.27) using the Gauss theorem

$$(\partial_t\varphi_\varepsilon, v) + (\nabla\varphi_\varepsilon, \nabla v) + (\beta_\varepsilon(\varphi_\varepsilon), v) = (-\pi(\varphi_\varepsilon) + \gamma\partial_t w_\varepsilon, v). \tag{4.5}$$

At this point, since $\varphi_\varepsilon \in H^1(0,T;V)$ and the considerations on the weak-derivability of $\beta_\varepsilon(\varphi_\varepsilon)$, $\pi(\varphi_\varepsilon)$ and $\partial_t w_\varepsilon$ remain valid, $\partial_t\varphi_\varepsilon$ is derivable with respect to time. We derive the above equation finding

$$(\partial_t^2\varphi_\varepsilon, v) + (\nabla\partial_t\varphi_\varepsilon, \nabla v) + (\beta_\varepsilon'(\varphi_\varepsilon)\partial_t\varphi_\varepsilon, v) = (\tilde{g}_4, v), \qquad \forall v \in V, \text{ a.e. in } (0,T),$$

where $\tilde{g}_4$ is defined in the same way as g_4 and it satisfies $\|\tilde{g}_4\|_{L^2(0,T;H)} \leq C(1+\rho^{1/2})$. We take $v = \partial_t \varphi_\varepsilon$ and after some calculations carried out as in the former part we arrive at

$$\frac{1}{2}\|\partial_t \varphi_\varepsilon(t)\|_H^2 + \int_0^t \|\nabla \partial_t \varphi_\varepsilon\|_H^2 \leq \frac{1}{2}\|\partial_t \varphi_\varepsilon(0)\|_H^2 + C(1+\rho) + \int_0^t \|\partial_t \varphi_\varepsilon\|_H^2.$$

Now, using the hypothesis $\beta^\circ(\varphi_0) \in H$ we deduce that

$$\|\beta_\varepsilon(\varphi_0)\|_H \leq \|\beta^\circ(\varphi_0)\|_H \leq C.$$

Hence, arguing as in equation (4.2), we have that $\|\partial_t \varphi_\varepsilon(0)\|_H \leq C$ and the Gronwall lemma allows us to deduce that

$$\|\varphi_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \leq C(1+\rho^{1/2}). \tag{4.6}$$

By comparison, we find $\|-\triangle \varphi_\varepsilon + \xi_\varepsilon\|_{L^\infty(0,T;H)} \leq C(1+\rho^{1/2})$, thus, by the argument used in § 3.3.2, we conclude that $\triangle \varphi_\varepsilon, \xi_\varepsilon \in L^\infty(0,T;H)$ and that

$$\|\triangle \varphi_\varepsilon\|_{L^\infty(0,T;H)} + \|\xi_\varepsilon\|_{L^\infty(0,T;H)} \leq C(1+\rho^{1/2}). \tag{4.7}$$

Chapter 5

Continuous dependence of the solutions

In order to simplify the notation, we let $\vartheta_0 = \vartheta_{0,1} - \vartheta_{0,2}$ and analogously we define w_0, φ_0, f, w, φ, ξ and σ. In this proof, C denotes a time-to-time-different, positive, large-enough constant independent of the just-said data and of ρ.

It is clear that

$$(w_t + l\varphi)_t - \kappa \triangle w_t - \tau \triangle w + \rho\sigma = f, \tag{5.1}$$

$$\varphi_t - \triangle\varphi + \xi + \pi(\varphi_1) - \pi(\varphi_2) = \gamma w_t. \tag{5.2}$$

We multiply equations (5.1) and (5.2) by $(w_t + l\varphi)$ and $\kappa l^2\varphi$ respectively, sum up and integrate over Ω obtaining

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t + l\varphi\|_H^2 + \kappa\|\nabla w_t\|_H^2 + \kappa l(\nabla w_t, \nabla\varphi) + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_H^2 \\
&+ \tau l(\nabla w, \nabla\varphi) + \rho(\sigma, w_t + l\varphi)_H + \frac{\kappa l^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_H^2 \\
&+ \kappa l^2\|\nabla\varphi\|_H^2 + \kappa l^2(\xi, \varphi)_H + \kappa l^2(\pi(\varphi_1) - \pi(\varphi_2), \varphi)_H \\
&= (f, w_t) + l(f, \varphi) + \gamma\kappa l^2(w_t, \varphi).
\end{aligned}$$

We rearrange and we use the Lipschitz-continuity of π, equations (2.12) and (2.14), and the monotonicity of Sign and β to infer that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t + l\varphi\|_H^2 + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_H^2 + \frac{\kappa l^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_H^2 \\
&+ \kappa(\|\nabla w_t\|_H^2 + l(\nabla w_t, \nabla\varphi) + l^2\|\nabla\varphi\|_H^2) \\
&\leq (f, w_t) + l(f, \varphi) + \gamma\kappa l^2(w_t, \varphi) + C\|\varphi\|_H^2 - \tau l(\nabla w, \nabla\varphi).
\end{aligned}$$

At this point, we use Young inequality and the fact that

$$l(\nabla w_t, \nabla\varphi) \geq -\frac{1}{2}(\|\nabla w_t\|_H^2 + l^2\|\nabla\varphi\|_H^2)$$

to deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t + l\varphi\|_H^2 + \frac{\kappa l^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_H^2 + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|_H^2 \\
&+ \frac{\kappa}{2}(\|\nabla w_t\|_H^2 + l^2\|\nabla\varphi\|_H^2) \\
&\leq C\|f(t)\|_H^2 + C(\|w_t\|_H^2 + \|\varphi\|_H^2) + \frac{\kappa l^2}{4}\|\nabla\varphi\|_H^2 + \frac{\tau^2}{\kappa}\|\nabla w\|_H^2.
\end{aligned}$$

We integrate between 0 and t

$$\begin{aligned}
&\frac{1}{2}\|w_t(t) + l\varphi(t)\|_H^2 + \frac{\kappa l^2}{2}\|\varphi(t)\|_H^2 + \frac{\tau}{2}\|\nabla w(t)\|_H^2 \\
&+ \frac{\kappa}{2}\int_0^t \|\nabla w_t\|_H^2 + \frac{\kappa l^2}{4}\int_0^t \|\nabla\varphi\|_H^2 \\
&\leq C\|f\|_{L^2(0,T;H)}^2 + \frac{1}{2}\|\vartheta_0 + l\varphi_0\|_H^2 + \frac{\kappa l^2}{2}\|\varphi_0\|_H^2 + \frac{\tau}{2}\|\nabla w_0\|_H^2 \\
&+ C\int_0^t (\|w_t\|_H^2 + \|\varphi\|_H^2) + \frac{\tau^2}{\kappa}\int_0^t \|\nabla w\|_H^2.
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
&C^{-1}(\|w_t(t)\|_H^2 + \|\varphi(t)\|_H^2) \leq \|w_t(t) + l\varphi(t)\|_H^2 + \kappa l^2\|\varphi(t)\|_H^2, \\
&\|\vartheta_0 + l\varphi_0\|_H^2 + \kappa l^2\|\varphi_0\|_H^2 \leq C(\|\vartheta_0\|_H^2 + \|\varphi_0\|_H^2),
\end{aligned}$$

and so we can apply the Gronwall Lemma finding

$$\begin{aligned}
&\|w_t\|_{L^\infty(0,T;H)} + \|\nabla w\|_{L^\infty(0,T;H)} + \|\nabla w_t\|_{L^2(0,T;H)} \\
&+ \|\varphi\|_{L^\infty(0,T;H)} + \|\nabla\varphi\|_{L^2(0,T;H)} \\
&\leq C(\|f\|_{L^2(0,T;H)} + \|\vartheta_0\|_H + \|\varphi_0\|_H + \|\nabla w_0\|_H).
\end{aligned}$$

This implies, as $w_0, \varphi_0 \in V$, that

$$\begin{aligned}
&\|w\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\
&\leq C(\|f\|_{L^2(0,T;H)} + \|\vartheta_0\|_H + \|w_0\|_V + \|\varphi_0\|_H),
\end{aligned} \tag{5.3}$$

and the proof is complete.

We conclude this chapter by giving the proof of Corollary 2.5. Assume that $(w_i, \varphi_i, \xi_i, \sigma_i)$, $i = 1, 2$, are two solutions given by the existence Theorem 2.2. Since $l = \alpha$ we can apply the just-proven Theorem 2.4 with $(\vartheta_{0,i}, w_{0,i}, \varphi_{0,i}, f_i) = (\vartheta_0, w_0, \varphi_0, f)$. Hence, by the equation above we deduce that $w_1 = w_2$ and $\varphi_1 = \varphi_2$. By comparison, we conclude that $\sigma_1 = \sigma_2$ and $\xi_1 = \xi_2$.

Chapter 6

Existence of sliding modes

This chapter is devoted to prove Theorem 2.6. Like in Section 3.5, we will use the more regular, approximated solutions and we will later take a limit as $\varepsilon \to 0$. Before going through the proof of the Theorem, we prove the following lemma.

Lemma 6.1. Let $T, M > 0$,[1] $\psi_0 \geq 0$, and let $\psi : [0, T] \to \mathbb{R}$ be an a non-negative, absolutely continuous function with $\psi(0) = \psi_0$. Let A be the set

$$A := \{t \in [0, T] : \psi(t) > 0\}. \tag{6.1}$$

If $\psi'(t) \leq -M$ a.e. in A, then the following conclusions hold true.

1) If $\psi_0 = 0$, then $\psi \equiv 0$.

2) If $M > \psi_0/T$, then there exist a time $T^* \in (0, T)$ such that

$$T^* \leq \frac{\psi_0}{M} < T, \tag{6.2}$$

as well as the function ψ is strictly decreasing in $[0, T^*)$ and vanishes in $[T^*, T]$.

Proof. 1) Suppose on the contrary that A is non empty. Let $B = (a, b)$ be a connected component of A. The function ψ restricted to B is strictly decreasing. Indeed, if $a < t_0 < t_1 < b$, we have that

$$\psi(t_1) - \psi(t_0) = \int_{t_0}^{t_1} \psi'(s)\mathrm{d}s \leq -M(t_1 - t_0) < 0.$$

[1]The first part of this lemma is still working if $M = 0$.

We now take the limit as $t_0 \to a$ obtaining

$$\psi(t_1) \leq \lim_{t_0 \to a} \psi(t_0) = \psi(a) = 0,$$

which is a contradiction for we assumed $\psi(t_1) > 0$.

2) We may assume $\psi_0 > 0$, because the case $\psi_0 = 0$ follows directly from the former part with $T^* = 0$. We define T^* as

$$T^* := \sup\{t \in (0,T) : \psi(s) > 0 \text{ for all } s \in (0,t)\}. \tag{6.3}$$

By continuity of ψ, T^* is well-defined and greater than 0. Moreover, the interval $[0, T^*)$ is contained in A, hence we have that

$$\psi(T^*) - \psi(0) = \int_0^{T^*} \psi'(t)\mathrm{d}t \leq -MT^*,$$

thus

$$T^* \leq \frac{\psi_0 - \psi(T^*)}{M} \leq \frac{\psi_0}{M} < T.$$

Note that ψ is strictly decreasing in $[0, T^*)$ for what we have proven in 1). It is clear that $\psi(T^*) = 0$. Indeed, if on the contrary $\psi(T^*) > 0$, then $\psi > 0$ in $[0, T^* + \varepsilon)$ for a small ε and the supremum in definition (6.3) fails. Finally, we define $\delta : [0, T - T^*] \to [0, +\infty)$ as $\delta(t) = \psi(t + T^*)$ and we use the first part of the lemma, deducing $\delta = 0$, thus $\psi(t) = 0$ for all $t \in [T^*, T]$. □

We define for $\varepsilon > 0$, $\eta_\varepsilon, g_{5,\varepsilon} : [0,T] \to H$ as

$$\begin{aligned} \eta_\varepsilon &:= \partial_t w_\varepsilon + \alpha\varphi_\varepsilon - \eta^*, \\ g_{5,\varepsilon} &:= \tau \triangle w_\varepsilon - \kappa\alpha \triangle \varphi_\varepsilon + (\alpha - l)\partial_t \varphi_\varepsilon - \kappa \triangle \eta^* + f. \end{aligned}$$

Analogously, we define $\eta = \partial_t w + \alpha\varphi - \eta^*$. Because of the proofs of Theorems 2.2 and 2.3, we infer that $g_{5,\varepsilon} \in L^\infty(0,T;H)$ and

$$\|g_{5,\varepsilon}\|_{L^\infty(0,T;H)} \leq C_5(1 + \rho^{1/2}),$$

where the constant C_5 is defined in equation (2.24). Recalling that $\sigma_\varepsilon = \mathrm{Sign}_\varepsilon(\eta_\varepsilon)$ we rewrite equation (3.26) as

$$(\partial_t \eta_\varepsilon - \kappa \triangle \eta_\varepsilon + \rho\sigma_\varepsilon, v) = (g_{5,\varepsilon}, v) \qquad \forall v \in H, \text{ a.e. in } (0,T).$$

We take $v = \sigma_\varepsilon$ in the above equation and integrate between t and $t + h$ (with $h \in (0, T - t)$) obtaining

$$\int_t^{t+h} (\partial_t \eta_\varepsilon, \sigma_\varepsilon) + \kappa \int_t^{t+h} (\nabla \eta_\varepsilon, \nabla \sigma_\varepsilon) + \rho \int_t^{t+h} \|\sigma_\varepsilon\|_{H,\varepsilon}^2 = \int_t^{t+h} (g_{5,\varepsilon}, \sigma_\varepsilon).$$

Owing to equation (3.8) we deduce

$$(\partial_t \eta_\varepsilon, \sigma_\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}t}\|\eta_\varepsilon\|_{H,\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\|\eta_\varepsilon(t)\|_H} \min\{s/\varepsilon, 1\}\mathrm{d}s.$$

In view of (3.6), we have that

$$\int_t^{t+h} (\nabla\eta_\varepsilon, \nabla\sigma_\varepsilon) = \int_t^{t+h} \frac{\|\nabla\eta_\varepsilon\|_H^2}{\max\{\varepsilon, \|\eta_\varepsilon\|_H\}} \geq 0.$$

Finally, as $\|\sigma_\varepsilon\|_H \leq 1$ a.e. in $(0,T)$, we have that

$$\int_t^{t+h} (g_{5,\varepsilon}, \sigma_\varepsilon) \leq hC_5(1+\rho^{1/2}).$$

Putting everything together we obtain

$$\int_0^{+\infty} \mathbf{1}_{[\|\eta_\varepsilon(t+h)\|_H, \|\eta_\varepsilon(t)\|_H]} \min\{s/\varepsilon, 1\}\mathrm{d}s + \rho\int_t^{t+h} \|\sigma_\varepsilon\|_{H,\varepsilon}^2 \leq hC_5(1+\rho^{1/2}), \tag{6.4}$$

where the meaning of $\mathbf{1}_{[a,b]}$ is the following: if $a \leq b$, then $\mathbf{1}_{[a,b]}$ is the characteristic function of the interval $[a,b]$; if $a > b$, then $\mathbf{1}_{[a,b]}$ means minus the characteristic function of $[b,a]$. We remark that η_ε converges in $C^0([0,T];H)$ as $\varepsilon \to 0$ along a subsequence (cf. (3.24)–(3.25), which can be replicated for the limit as $\varepsilon \to 0$). Hence, we have the following convergence for the functions

$$\mathbf{1}_{[\|\eta_\varepsilon(t+h)\|_H, \|\eta_\varepsilon(t)\|_H]} \to \mathbf{1}_{[\|\eta(t+h)\|_H, \|\eta(t)\|_H]} \qquad \text{a.e. in } (0,T).$$

At this point, since all the involved functions are bounded, we take the inferior limit in (6.4) as $\varepsilon \to 0$ and we use Lebesgue dominate convergence theorem, property (3.34), and the weak lower semicontinuity of norms obtaining

$$\|\eta(t+h)\|_H - \|\eta(t)\|_H + \rho\int_t^{t+h} \|\sigma\|_H^2 \leq hC_5(1+\rho^{1/2}).$$

We divide by h and we take the limit as $h \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\eta(t)\|_H) + \rho\|\sigma(t)\|_H^2 \leq C_5(1+\rho^{1/2}) \qquad \text{for a.a. } t \in (0,T).$$

We introduce the function $\psi(t) = \|\eta(t)\|_H$ and the quantity

$$M(\rho) = \rho - C_5 - C_5\rho^{1/2}.$$

We also set (see (2.25)) $\psi_0 = \|\vartheta_0 + \alpha - \eta^*\|_H$. The inequality above implies

$$\psi'(t) \leq -M(\rho), \qquad \text{for a.a. } t \text{ in } \{t : \psi(t) > 0\}. \tag{6.5}$$

Using the Young inequality we obtain

$$M(\rho) \geq \frac{\rho}{2} - C_5 - \frac{C_5^2}{2}. \tag{6.6}$$

Thus, if we chose

$$\rho^* = 2\left(\frac{\psi_0}{T} + C_5 + \frac{C_5^2}{2}\right),$$

then for every $\rho > \rho^*$, $M(\rho) > \psi_0/T$.

Finally we can use Lemma 6.1, that guarantees the existence of $T^* < T$ such that ψ vanishes in $[T^*, T]$, i.e. the thesis. Moreover, the second part of the Lemma and equation (6.6) lead to

$$T^* \leq \frac{2\psi_0}{\rho - 2C_5 - C_5^2} < T.$$

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