On the geometry of synthetic null hypersurfaces and the Null Energy Condition

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Definition. A smooth space-time satisfies the Null Energy Condition (NEC), iff

$$\mathrm{Ric}(v,v) \geq 0, \quad \forall v \in TM, \text{ with } g(v,v) = 0$$

General: expected to be satified by all forms of matter.

Consequences:

- Hawking's Area Theorem
- Penrose's Singularity Theorem

Main object: null hypersufaces. $H \subset M$ is a null hypersurface if $g|_{TH}$ is degenerate.

Why non-smooth space-times?

Many physically relevant space-times have discontinuous energy-momentum tensor, leading to Lorentzian metrics of low regularity.

- Impulsive gravitational waves (Penrose 1972)
- Matched space-times modeling interior and exterior of stars.

Singularity theorems (Penrose, Hawking, ...): under suitable assumptions $\Rightarrow \exists$ a causal (light-like for Penrose, time-like for Hawking) geodesic γ whose maximal domain $\subsetneq \mathbb{R}$, i.e, it is not complete.

FACT: classical Thms require smoothness (say C^2).

Classical Thms do not prevent to extend the spacetime (M, g) to a spacetime of lower regularity $\left(\widetilde{M}, \tilde{g}\right)$.

WAY OUT: generalize singularity thms to non-smooth (C^1, $C^{0,1}$, C^0, ...) spacetimes

CHALLENGE: give a meaning of $\mathrm{Ric} \geq K$.

Ricci curvature in non-smooth space-times

- If $g, g^{-1} \in L^\infty_{\mathrm{loc}} \cap W^{1,2}$, then one can understand Ric in the distributional sense (Geroch–Traschen 1987).
- Works only if smooth charts are available. What it charts are not available?
- Displacement convexity of the entropy using Optimal Transport.

- Characterization of $\mathrm{Ric} \geq K \in \mathbb{R}$, via displacement convexity for smooth Riemannian manifold (McCann, Otto–Villani, von Renesse–Sturm, ...).

  $\rightarrow$ starting point for the theory of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ and the $\mathsf{CD}(K, N)$ condition (Lott–Sturm–Villani).

  Synthetic data: metric + reference measure

- Characterization of $\mathrm{Ric}(v, v) \geq K g(v, v)$, for any v time-like (McCann, Mondino–Suhr).

  $\rightarrow$ starting point for the theory of Lorentzian pre-length space (Kunzinger–Sämann) and $\mathsf{TCD}(K, N)$ condition (Cavalletti–Mondino).

  Synthetic data: causal relations + time separation + reference measure

Null Energy Condition: previous approaches

- McCann (2023): Characterization of NEC as limit of Ricci lower bounds on time-like directions (for which a synthetic definition is available)
  - **Pros**: Working in non-smooth Lorentzian pre-length space
  - **Cons**: Not stable under convergence. Open problem to draw applications
- Ketterer (2023): Characterization of NEC by means of convexity of the entropy along Optimal Transport of measures of concentrated on sets of co-dimension 2
  - **Pros**: fruitful for applications, e.g., OT proof of Hawking's Area Theorem and of Penrose's singularity theorem.
  - **Cons**: characterization is heavily relaying on smoothness; not clear how to generalize to the non-smooth setting.

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NEC and OT along null hypersurfaces

Let (M, g) be a smooth Lorentzian manifold

Definition. A hypersurface $H \subset M$ is null

- iff the restriction of g to TH is degenerate
- iff $\exists L \in \Gamma(TH)$, $L \neq 0$, a vector field such that $g(L, L) = 0$

We may assume $\nabla_L L = 0$

FACTS:

1. H is foliated by geodesics
2. causal curves in H are pre-geodesics (parametrizations of geodesics)
3. no canonical distance on H

CHALLENGES:

1. no purely-metric characterization of geodesics
2. the determinant of g on H is null, thus no volume form.

$\rightarrow$ We can use the rigged volume Vol_L (albeit non-unique?!)

Geodesics = optimal way to transport *points*

Optimal Transport = optimal way to transport *probability measures*

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Let μ_0, μ_1 be two probability measures. The (dynamical) optimal transport problem is

$$\sup_{\substack{\nu \in \mathcal{P}(\mathrm{Caus}(M)): \\ (e_i)_\sharp \nu = \mu_i, i=0,1}} \int_{C([0,1];M)} \int_0^1 |\dot{\gamma}_t|^p \, \mathrm{d}t \, \nu(\mathrm{d}\gamma),$$

where $e_t(\gamma) = \gamma(t)$ is the evaluation map, $p \in (0,1]$.

$\mathrm{OptCaus}(\mu_0, \mu_1) :=$ {optimizers} $\subset \mathcal{P}(\mathrm{Caus}(M))$

Remark. On a null hypersurface, $|\dot{\gamma}| = 0$, thus *optimal transport* = *admissible transport*.
Optimizers are not concentrated on geodesics (in general).

Theorem (Cavalletti–M.–Mondino). Let (M^n, g) be a strongly causal, C^2 manifold. TFAE

1. $\mathrm{Ric}(v,v) \geq 0$, for all $v \in TM$ light-like
2. for every $H \subset M$ null, for every $\mu_0, \mu_1 \in \mathcal{P}(H)$, such that $\mathrm{OptGeo}(\mu_0, \mu_1) \neq \emptyset$, there exists $\nu \in \mathrm{OptCaus}(\mu_0, \mu_1)$, such that

$$U_{n-1}(\mu_t|\mathrm{Vol}_L) \geq (1-t)U_{n-1}(\mu_0|\mathrm{Vol}_L) + tU_{n-1}(\mu_1|\mathrm{Vol}_L),$$

where

$$\mu_t = (e_t)_\sharp \nu, \qquad U_N(\rho\mathfrak{m}|\mathfrak{m}) = \exp\left(-\frac{1}{N}\int \rho \log \rho \mathrm{d}\mathfrak{m}\right).$$

Remark. For the vast majority of $\nu \in \mathrm{OptCaus}(\mu_0, \mu_1)$, the inequality does not hold! $\mathrm{OptCaus}(\mu_0, \mu_1)$ is stable under reparameterization.

Synthetic null hypersurfaces

Definition (Causal space)**.** $(X, \leq, \ll)$ is *causal space* iff $\leq$ is a pre-order (transitive and reflexive) and $\ll$ is a transitive relation included in $\leq$.

$\leq$ is the causal relation; $\ll$ is the chronological relation

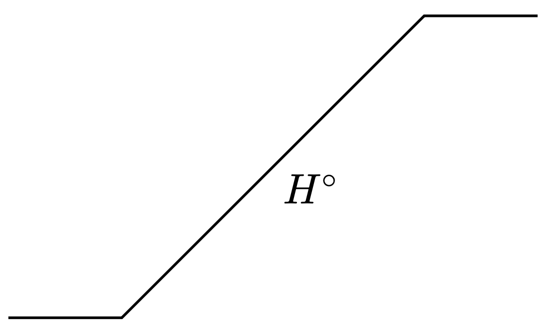
Causal curves are continuous. The set of causal curves will be denoted by Caus.

Remark. No time separation is used

Closed achronal sets are a synthetic replacement for null hypersurfaces.

Achronal boundaries $\partial I^+(S)$ are maximal closed achronal sets.

$H^\circ :=$ the achronal set without endpoints (i.e., maximal or minimal elements for $\leq$ in H)



Gauges select geodesics

Any causal curves in H is a pre-geodesic.

How to select geodesics?

Definition (Gauge)**.** A map $G : H^\circ \to \mathbb{R}$ is a gauge iff $\forall\gamma$ causal and injective in H°, $G\circ\gamma$ is strictly increasing and continuous;

Definition (G-causal)**.** $\gamma : [0,1] \to H^\circ$ is G-causal, iff it is causal and $G\circ\gamma$ is affine. $\mathrm{Caus}_G \subset \mathrm{Caus}$ is the set of G-causal curves.

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How to build a gauge in a smooth null hypersurface?

- $H \subset M$: smooth null hypersurface
- $L \in \Gamma(\mathcal{N}H)$: a null geodesic vector field (i.e., $\nabla_L L = 0$)
- $\Psi_L : \mathrm{Dom}(\Psi_L) \subset \mathbb{R} \times H \to H$: the flow of L
- $S \subset H$: a space-like, acausal, cross-section (i.e., $\Psi_L(\mathbb{R} \times S) = H$)

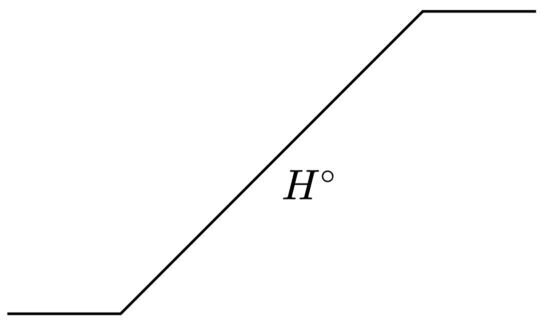
$G_{L,S}(x) = t$, iff $\exists ! y \in S$ such that $\Psi_L(t, y) = x$.

Proposition. Let $\gamma \in \mathrm{Caus}(H)$. TFAE:

- $\nabla_{\dot\gamma}\dot\gamma = 0$
- $\gamma \in \mathrm{Caus}_{G_{L,S}}$

Definition (Synthetic null hypersurface)**.** A triple $(H, G, \mathfrak{m})$ is a synthetic null hypersurface iff

- H is a closed achronal set
- $G : H^\circ \to \mathbb{R}$ is a gauge
- $\mathfrak{m} \in \mathcal{M}^+(H^\circ)$ is a Borel measure



Example. If $H \subset M$ is a smooth null hypersurface, $L \in \Gamma(\mathcal{N}H)$ is a null geodesic vector field, $S \subset H$ is a cross-section, then $\big(H, G_{L,S}, \mathrm{Vol}_L\big)$ is a synthetic null hypersurface.

The $\mathsf{NC}^e(N)$ Condition

$$\mathrm{OptCaus}(\mu_0,\mu_1) := \Big\{\nu \in \mathcal{P}(C([0,1];H)) : (e_i)_\sharp \nu = \mu_i, i = 0,1, \nu(\mathrm{Caus}) = 1\Big\}$$

$$\mathrm{OptCaus}_G(\mu_0,\mu_1) := \{\nu \in \mathrm{OptCaus}(\mu_0,\mu_1) : \nu(\mathrm{Caus}_G) = 1\}$$

Definition (NC$^e(N)$ condition)**.** Let $N > 1$. We say that $(H, G, \mathfrak{m})$ satisfyies NC$^e(N)$, iff: $\forall \mu_0, \mu_1 \in \mathcal{P}(H^\circ)$, such that $\mathrm{OptCaus}(\mu_0,\mu_1) \neq \emptyset$, there exists $\nu \in \mathrm{OptCaus}_G(\mu_0,\mu_1)$, such that

$$U_{N-1}\Big((e_t)_\sharp \mu|\mathfrak{m}\Big) \geq (1-t)\, U_{N-1}\Big((e_0)_\sharp \mu|\mathfrak{m}\Big) + t\, U_{N-1}\Big((e_1)_\sharp \mu|\mathfrak{m}\Big),$$

where

$$\mu_t := (e_t)_\sharp \nu, \qquad U_N(\rho\mathfrak{m}|\mathfrak{m}) := \exp\left(-\frac{1}{N}\int_H \rho \log(\rho)\,\mathrm{d}\mathfrak{m}\right)$$

Theorem (Cavalletti–M.–Mondino)**.** Let (M^n, g) be a Lorenzian manifold satisfying NEC. Let $H \subset M$ be a null hypersurface. Then the synthetic null hypersurface $\big(H, G_{L,S}, \mathrm{Vol}_L\big)$ is NC$^e(n)$.

Problem: $G_{L,S}$ and Vol_L are not uniquely determined

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Definition (Transverse function). A function $f : H^\circ \to \mathbb{R}$ is transverse if it is constant along causal curves.

If $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{M}^+(H)$, we shall write $\mathfrak{m}_1 \sim \mathfrak{m}_2$ iff $\mathfrak{m}_1 = f\mathfrak{m}_2$, for some $f > 0$ transverse.

If $G_1, G_2 : H^\circ \to \mathbb{R}$ are two gauges, we will write $G_1 \sim G_2$, if $G_1 = fG_2 + h$.

Theorem (Cavalletti–M.–Mondino). Let $(H, G_1, \mathfrak{m}_1), (H, G_2, \mathfrak{m}_2)$ be two synthetic null hypersurfaces. Assume that $G_1 \sim G_2$ and $\mathfrak{m}_1 \sim \mathfrak{m}_2$. Then

$$(H, G_1, \mathfrak{m}_1) \text{ is } \mathsf{NC}^e(N) \qquad \Longleftrightarrow \qquad (H, G_2, \mathfrak{m}_2) \text{ is } \mathsf{NC}^e(N)$$

Hawking's Area Theorem

Define $\Psi_G : \mathrm{Dom}(\Psi_G) \subset \mathbb{R} \times H^\circ \to H$, so that

$$t \mapsto \Psi_G(t,x) \in \mathrm{Caus}(H) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} G(\Psi_G(t,x)) = 1$$

Given S in $(H, G, \mathfrak{m})$, we define

$$S_\varepsilon^+ := \{x \in H^\circ : \Psi_G(x,-r) \in S, r \in (0,\varepsilon)\}.$$

Definition. The *(future) Minkowski content* of S is

$$\mathfrak{m}_G^+(S) := \limsup_{\varepsilon \to 0^+} \frac{\mathfrak{m}(S_\varepsilon^+)}{\varepsilon}.$$

Definition. A closed achronal set H is future complete for a gauge G iff

$$\sup_{y \in J^+(z) \cap H^\circ} G(y) = \infty, \qquad \forall z \in H^\circ.$$

Theorem (Cavalletti, M., Mondino)**.** Let $(H, G, \mathfrak{m})$ be a synthetic null hypersurface, satisfying the $\mathsf{NC}^e(N)$ condition. Assume H to be future complete for G and that $\mathrm{supp}(\mathfrak{m}) = H$. Let S_1, S_2 be two acausal sets with $S_1 \subset J^-(S_2)$. Then

$$\mathfrak{m}^+_G(S_1) \leq \mathfrak{m}^+_G(S_2),$$

where the value $+\infty$ is admitted.

Remark. Let G_1 and G_2 be two gauges for H. If H is future complete for G_1, then so it is also for G_2.

Theorem. Let H be a null non-branching closed achronal set and $S \subset H$ be an acausal set. Let $\mathfrak{m}_i \in \mathcal{M}^+(H)$ be a measure and G_i be a gauge, $i = 1, 2$. Assume that

- $G_1 = fG_2 + h \quad \text{and} \quad \mathfrak{m}_1 = \frac{1}{f}\mathfrak{m}_2 \qquad \text{for some } f, g \text{ transverse}, f > 0,$
- $(H, G_1, \mathfrak{m}_1)$ satisfies $\mathsf{NC}^e(N)$ (thus also $(H, G_2, \mathfrak{m}_2)$ does),
- H is future complete for G_1 (thus also for G_2),
- $(\mathfrak{m}_i)^+_{G_i}(S) < \infty$, $i = 1, 2$.

Then

$$(\mathfrak{m}_1)^+_{G_1}(S) = (\mathfrak{m}_2)^+_{G_2}(S).$$

Penrose's Singularity Theorem

Recap of the smooth setting

Theorem (Penrose)**.** Let (M, g) be a C^∞ Lorentzian manifold. Assume:

- M satisfies NEC
- M is future null complete
- There exists $S \subset M$ be a compact, space-like, achronal, future-converging surface with $\operatorname{codim} S = 2$.

Then S is future-trapped set.

future-converging surface $= H^+, H^- < \theta < 0$, where $H^\pm$ is the inward/outward mean curvature

future-trapped set $= \partial I^+(S)$ is compact

future null complete $= \forall \gamma : [0, b) \to M$ maximally-defined (on the right) null geodesic, $b = \infty$

Theorem (Penrose)**.** The following cannot be true at the same time:

- There exists a non-compact Cauchy hypersurface;
- There exists a future-trapped set.

- Extended for $C^{1,1}$ metrics by Kunzinger–Steinbauer–Vickers (2015), and for C^1 metrics by Graf (2020). NEC is understood in the distributional sense and geodesics in ODE sense.
- The second part was extended to C^0 metrics by Ling (2020) and closed-cones structures by Minguzzi (2019)

In order to lower the regularity to the synthetic setting, we need to understand what is *future null completeness* and *future converging surface*.

Definition.

A strongly causal, Lorentzian manifold (M, g) is *future weakly null complete*, iff $\forall \gamma : [0, b) \to M$ maximally-defined (on the right) null geodesic (in the differential sense), either $b = \infty$ or $\gamma_0 \ll \gamma_t$, for some $t > 0$.

We can define also *past weak null completeness*.

Proposition (Cavalletti, M., Mondino)**.**

Let (M, g) be a strongly causal, causally closed C^2 Lorentzian manifold. TFAE:

- M is weakly future null complete.
- For all $A \subset M$ spacelike achronal submanifold, $\partial I^+(A)$ admits a proper natural gauge, which can be extended continuously to A.
- The same, with A being a singleton.

Properness of the gauge is the synthetic counterpart of completeness!

Definition. Let $(H, G, \mathfrak{m})$ be a $\mathsf{NC}^e(N)$ synthetic null hypersurface, with $H = \partial I^+(S)$, S achronal.

We say that S is $(G, \mathfrak{m})$-future converging, if there exists $\theta < 0$, such that

$$\limsup_{\varepsilon \to 0^+} \frac{\mathfrak{m}(A_\varepsilon^+) - \varepsilon \mathfrak{m}_G^+(A)}{\varepsilon^2/2} \leq \theta \mathfrak{m}_G^+(A), \qquad \forall A \subset S$$

Theorem (Cavalletti–M.–Mondino)**.**

Let $(H, G, \mathfrak{m})$ be a $\mathsf{NC}^e(N)$ synthetic null hypersurface, with $H = \partial I^+(S)$, S achronal and compact.

Assume that $\mathrm{supp}(\mathfrak{m}) = H$ and that

$$\inf_t G(\Psi_G(x,t)) = 0, \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in H^\circ.$$

If S is $(G, \mathfrak{m})$-future converging, then $\|G\|_{L^\infty(\mathfrak{m})} \leq C(\theta) < \infty$.

In particular, if the gauge G is proper, then H compact, i.e., S is future trapped.

Remark. If we replace G with $G' := fG$ and $\mathfrak{m}$ with $\mathfrak{m}' := \frac{1}{f}\mathfrak{m}$, for some f transverse, with $\frac{1}{C} < f < C$, then the assumptions still stand.

In particular S is $(G, \mathfrak{m})$-future converging, iff it is $(G', \mathfrak{m}')$-future converging.

Thank you for the attention!