On the Geometry of Synthetic Null Hypersurfaces and the Null Energy Condition

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The Null Energy Condition

Definition. A smooth Lorentzian manifold (M, g) satisfies the **Null Energy Condition (NEC)** iff

$$\mathrm{Ric}(v, v) \geq 0, \quad \forall v \in TM \ \text{ with } \ g(v, v) = 0.$$

Expected to hold for all classical forms of matter. **Consequences:**

- Hawking's Area Theorem
- Penrose's Singularity Theorem

Main object. A hypersurface $H \subset M$ is a **null hypersurface** if $g|_{TH}$ is degenerate, or equivalently if there exists $L \in \Gamma(TH)$, $L \neq 0$, with $g(L, L) = 0$.

Facts. H is foliated by null geodesics; there is no canonical distance nor canonical measure on H.

The **rigged volume** Vol_L (attached to a null geodesic vector field L, with $\nabla_L L = 0$) serves as reference measure.

Why Non-Smooth Space-Times?

Physically relevant space-times carry a non-smooth structure:

- **Impulsive gravitational waves**
- **Matched space-times** modelling the interior/exterior of stars

Classical singularity theorems (Penrose, Hawking) require smoothness (C^2), so they do not prevent extending (M, g) to a space-time of lower regularity $\big(\widetilde{M}, \tilde{g}\big)$.

Goal. Generalise singularity theorems to metrics of class C^1, $C^{0,1}$, C^0, or even to metric/causal spaces.

Challenge. Give a rigorous synthetic meaning to $\mathrm{Ric}(v, v) \geq 0$ for null v.

Synthetic Ricci Curvature via Optimal Transport

Riemannian case. The characterisation of $\mathrm{Ric} \geq K$ via **displacement convexity** of entropy along W_2-geodesics (McCann, Otto–Villani, von Renesse–Sturm...) is the foundation of the $\mathsf{CD}(K, N)$ condition (Lott–Sturm–Villani) for metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$.

Lorentzian case. The analogous characterisation for **time-like** directions (McCann, Mondino–Suhr) led to the $\mathsf{TCD}(K, N)$ condition (Cavalletti–Mondino) for Lorentzian pre-length spaces.

This work. A synthetic characterisation of $\mathrm{Ric}(v, v) \geq 0$ for **null** v, via optimal transport along null hypersurfaces.

NEC as Displacement Convexity

Optimal transport on a null hypersurface. Since $|\dot\gamma| = 0$ on H, every transport plan is optimal: **optimal transport = admissible transport**.

For $\mu_0, \mu_1 \in \mathcal{P}(H)$ define

$$\mathrm{OptCaus}(\mu_0, \mu_1) := \Big\{\nu \in \mathcal{P}(\mathrm{Caus}(H)) : (e_i)_\sharp \nu = \mu_i, i = 0, 1\Big\},$$

where $e_t(\gamma) := \gamma(t)$ $\mathrm{Caus}(H) = \{\text{causal curves in } H\}$.

Theorem ([1]). Let (M^n, g) be strongly causal and C^2. The following are equivalent:

1. $\mathrm{Ric}(v, v) \geq 0$ for all light-like $v \in TM$.
2. For every null $H \subset M$, every $\mu_0, \mu_1 \in \mathcal{P}(H)$ with $\mathrm{OptGeo}(\mu_0, \mu_1) \neq \emptyset$, there exists $\nu \in \mathrm{OptCaus}(\mu_0, \mu_1)$ such that

$$t \mapsto U_{n-1}\Big((e_t)_\sharp \nu \mid \mathrm{Vol}_L\Big) \ \text{ is convex}$$

where $U_N(\rho\mathfrak{m}|\mathfrak{m}) := \exp(-\frac{1}{N}\int \rho \log \rho, \mathrm{d}\mathfrak{m})$.

Synthetic Null Hypersurfaces

Definition (Causal space). $(X, \leq, \ll)$ is a **causal space** if $\leq$ is a pre-order (causal relation) and $\ll$ is a transitive relation (chronological relation) with $\ll \subset \leq$.

Achronal sets. $H \subset X$ is achronal if no two points are chronologically related. Closed achronal sets replace null hypersurfaces in the non-smooth setting.

Definition (Gauge). $G : H \to \mathbb{R}$ is a **gauge** if $G \circ \gamma$ is strictly increasing and continuous for every causal injective curve γ in H.

$\mathrm{Caus}_G \subset \mathrm{Caus}$ is the set of G-causal curves, i.e., causal curves on which $G \circ \gamma$ is affine.

Definition (Synthetic null hypersurface). A triple $(H, G, \mathfrak{m})$ where H is a closed achronal set, $G : H \to \mathbb{R}$ a gauge, and $\mathfrak{m} \in \mathcal{M}^+(H)$.

Example (Smooth setting). Let H be a null and L be a null geodesic vector field (i.e., $g(L, L) = 0$ and $\nabla_L L = 0$). Ψ_L is the flow of L; $S \subset H$ is a space-like cross-section.

- $G_{L,S}(x) = t$ iff $\Psi_L(t, y) = x$ for a unique $y \in S$.
- The rigged volume Vol_L serves as reference measure.

If $H \subset M$ is smooth null, L a null geodesic vector field, S a cross-section, then $\big(H, G_{L,S}, \mathrm{Vol}_L\big)$ is a synthetic null hypersurface.

$\mathsf{NC}^e(N)$ Condition

Let $\mathrm{OptCaus}_G(\mu_0, \mu_1) := \{\nu \in \mathrm{OptCaus}(\mu_0, \mu_1) : \nu(\mathrm{Caus}_G) = 1\}$.

Definition ($\mathsf{NC}^e(N)$). $(H, G, \mathfrak{m})$ satisfies $\mathsf{NC}^{e(N)}$ (for $N > 1$) if for all $\mu_0, \mu_1 \in \mathcal{P}(H)$ with $\mathrm{OptCaus}(\mu_0, \mu_1) \neq \emptyset$, there exists $\nu \in \mathrm{OptCaus}_{G(\mu_0, \mu_1)}$ such that

$$t \mapsto U_{N-1}\Big((e_t)_\sharp \nu \mid \mathfrak{m}\Big) \ \text{ is convex.}$$

Theorem ([1,2]). If (M^n, g) satisfies NEC and $H \subset M$ is a null hypersurface, then $\big(H, G_{L,S}, \mathrm{Vol}_L\big)$ is $\mathsf{NC}^{e(n)}$.

Well-posedness. The gauge $G_{L,S}$ and volume Vol_L are not unique. Write $G_1 \sim G_2$ if $G_1 = fG_2 + h$ and $\mathfrak{m}_1 \sim \mathfrak{m}_2$ if $\mathfrak{m}_1 = f\mathfrak{m}_2$, for some $f > 0$, constant along causal curves.

Theorem ([2]). If $G_1 \sim G_2$ and $\mathfrak{m}_1 \sim \mathfrak{m}_2$, then

$$(H, G_1, \mathfrak{m}_1) \text{ is } \mathsf{NC}^{e(N)} \quad \iff \quad (H, G_2, \mathfrak{m}_2) \text{ is } \mathsf{NC}^{e(N)}.$$

Hawking's Area Theorem

Let $\Psi_G(x, r) = y$ if $G(y) - G(x) = r$ and $x \gneqq y$.

For $S \subset H$ acausal and $\varepsilon > 0$, let

$$S_\varepsilon^+ := \{x \in H : \Psi_G(x, -r) \in S, \ r \in (0, \varepsilon)\}.$$

Definition. The **(future) Minkowski content** of S is

$$\mathfrak{m}_G^+(S) := \limsup_{\varepsilon \to 0^+} \frac{\mathfrak{m}(S_\varepsilon^+)}{\varepsilon}.$$

Theorem ([2]). Let $(H, G, \mathfrak{m})$ be $\mathsf{NC}^e(N)$, future complete, with $\mathrm{supp}(\mathfrak{m}) = H$. If $S_1 \subset J^-(S_2)$ are acausal subsets of H, then $\mathfrak{m}_G^+(S_1) \leq \mathfrak{m}_G^+(S_2)$.

Remark. The Minkowski content $\mathfrak{m}_G^+(S)$ is independent of the choice of equivalent gauge–measure pair $(G, \mathfrak{m})$, whenever it is finite.